

ASYMPTOTIC BEHAVIOR OF THE TOTAL LENGTH OF EXTERNAL BRANCHES FOR BETA-COALESCENTS

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ABSTRACT. In this paper, we consider the Beta($2 - \alpha, \alpha$)-coalescents with $1 < \alpha < 2$ and study the moments of external branches, in particular the total external branch length $L_{ext}^{(n)}$ of an initial sample of n individuals. For this class of coalescents, it has been proved that $n^{\alpha-1}T^{(n)} \xrightarrow{(d)} T$, where $T^{(n)}$ is the length of an external branch chosen at random, and T is a known non negative random variable. We get the asymptotic behaviour of several moments of $L_{ext}^{(n)}$. As a consequence, we obtain that for Beta($2 - \alpha, \alpha$)-coalescents with $1 < \alpha < 2$, $\lim_{n \rightarrow +\infty} n^{3\alpha-5} \mathbb{E}[(L_{ext}^{(n)} - n^{2-\alpha} \mathbb{E}[T])^2] = \frac{((\alpha - 1)\Gamma(\alpha + 1))^2 \Gamma(4 - \alpha)}{(3 - \alpha)\Gamma(4 - 2\alpha)}$.

1. INTRODUCTION

1.1. Motivation. In a Wright-Fisher haploid population model with size N , we sample n individuals at present from the total population, and look backward to see the ancestral tree until we get the most recent common ancestor (MRCA). If time is well rescaled and the size N of population becomes large, then the genealogy of the sample of size n converges weakly to the Kingman n -coalescent (see [32],[33]). During the evolution of the population, mutations may occur. We consider the infinite sites model introduced by Kimura [31]. In this model, each mutation is produced at a new site which is never seen before and will never be seen in the future. The neutrality of mutations means that all mutants are equally privileged by the environment. Under the infinite sites model, to detect or reject the neutrality when the genealogy is given by the Kingman coalescent, Fu and Li[21] have proposed a statistical test based on the total mutation numbers on the external branches and internal branches. Mutations on external branches affect only single individuals, so in practice they can be picked out according to the model setting. In this test, the ratio $L_{ext}^{(n)}/L^{(n)}$ between the total external branch length $L_{ext}^{(n)}$ and the total length $L^{(n)}$ measures in some sense the weight of mutations occurred on external branches among all. It then makes the study of these quantities relevant.

For many populations, Kingman coalescent describes the genealogy quite well. But for some others, when descendants of one individual can occupy a big ratio of the next generation with non-negligible probability, it is no more relevant. It is for example the case of some marine species (see [1], [9], [18], [23], [25]). In this case, if time is well rescaled and the size of population becomes large, the ancestral tree converges weakly to the Λ -coalescent which is associated with a finite measure Λ on $[0, 1]$. This coalescent allows multiple collisions. It has first been introduced by Pitman[37] and Sagitov[38]. Among Λ -coalescents, a special and important subclass is called Beta(a, b)-coalescents characterized by Λ being a Beta distribution $Beta(a, b)$. The most popular ones are those with parameters $2 - \alpha$ and α where $\alpha \in (0, 2)$.

Beta-coalescents arise not only in the context of biology. They also have connections with supercritical Galton-Watson process (see [39]), with continuous-state branching processes (see [6], [2], [19]),

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with continuous random trees (see [4]). If $\alpha = 1$, we recover the Bolthausen-Sznitman coalescent which appears in the field of spin glasses (see [8], [10]) and is also connected to random recursive trees (see [24]). The Kingman coalescent is also obtained from the $Beta(2 - \alpha, \alpha)$ -coalescent by letting α tend to 2.

For $Beta(2 - \alpha, \alpha)$ -coalescents with $1 < \alpha < 2$, a central limit theorem of the total external branch length $L_{ext}^{(n)}$ is known (see [30]). We should say that in this case, the moment method is not able to obtain the right convergence speed in the central limit theorem, which illustrates some limitations of moment calculations.

1.2. Introduction and main results. Let \mathcal{E} be the set of partitions of $\mathbb{N} := \{1, 2, 3, \dots\}$ and, for $n \in \mathbb{N}$, \mathcal{E}_n be the set of partitions of $\mathbb{N}_n := \{1, 2, \dots, n\}$. We denote by $\rho^{(n)}$ the natural restriction on \mathcal{E}_n : if $1 \leq n \leq m \leq +\infty$ and $\pi = \{A_i\}_{i \in I}$ is a partition of \mathbb{N}_m , then $\rho^{(n)}\pi$ is the partition of \mathbb{N}_n defined by $\rho^{(n)}\pi = \{A_i \cap \mathbb{N}_n\}_{i \in I}$. For a finite measure Λ on $[0, 1]$, we denote by $\Pi = (\Pi_t)_{t \geq 0}$ the Λ -coalescent process introduced independently by Pitman[37] and Sagitov[38]. The process $(\Pi_t)_{t \geq 0}$ is a càd-làg continuous time Markovian process taking values in \mathcal{E} with $\Pi_0 = \{\{1\}, \{2\}, \{3\}, \dots\}$. It is characterized by the càd-làg Λ n -coalescent processes $(\Pi_t^{(n)})_{t \geq 0} := (\rho^{(n)}\Pi_t)_{t \geq 0}$, $n \in \mathbb{N}$. For $n \leq m \leq +\infty$, we have $(\Pi_t^{(n)})_{t \geq 0} = (\rho^{(n)}\Pi_t^{(m)})_{t \geq 0}$ (where $\Pi^{(+\infty)} = \Pi$).

Let $\nu(dx) = x^{-2}\Lambda(dx)$. For $2 \leq a \leq b$, we set

$$\lambda_{b,a} = \int_0^1 x^{a-2}(1-x)^{b-a}\Lambda(dx) = \int_0^1 x^a(1-x)^{b-a}\nu(dx).$$

$\Pi^{(n)}$ is a Markovian process with values in \mathcal{E}_n , and its transition rates are given by: for $\xi, \eta \in \mathcal{E}_n$, $q_{\xi, \eta} = \lambda_{b,a}$ if η is obtained by merging a of the $b = |\xi|$ blocks of ξ and letting the $b - a$ others unchanged, and $q_{\xi, \eta} = 0$ otherwise. We say that a individuals (or blocks) of ξ have been coalesced in one single individual of η . Remark that the process $\Pi^{(n)}$ is an exchangeable process, which means that, for any permutation τ of \mathbb{N}_n , $\tau \circ \Pi^{(n)} \stackrel{(d)}{=} \Pi^{(n)}$.

The process $\Pi^{(n)}$ finally reaches one block. This final individual is called the most recent common ancestor (MRCA). We denote by $\tau^{(n)}$ the number of collisions it takes for the n individuals to be coalesced to the MRCA.

We define by $R^{(n)} = (R_t^{(n)})_{t \geq 0}$ the block counting process of $(\Pi_t^{(n)})_{t \geq 0}$: $R_t^{(n)} = |\Pi_t^{(n)}|$, which equals the number of blocks/individuals at time t . Then $R^{(n)}$ is a continuous time Markovian process taking values in \mathbb{N}_n , decreasing from n to 1. At state b , for $a = 2, \dots, b$, each of the $\binom{b}{a}$ groups with a individuals coalesces independently at rate $\lambda_{b,a}$. Hence, the time the process $(R_t^{(n)})_{t \geq 0}$ stays at state b is exponential with parameter:

$$(1) \quad g_b = \sum_{a=2}^b \binom{b}{a} \lambda_{b,a} = \int_0^1 (1 - (1-x)^b - bx(1-x)^{b-1})\nu(dx) = b(b-1) \int_0^1 t(1-t)^{b-2}\rho(t)dt,$$

where $\rho(t) = \int_t^1 \nu(dx)$. We denote by $Y^{(n)} = (Y_k^{(n)})_{k \geq 0}$ the discrete time Markov chain associated with $R^{(n)}$. This is a decreasing process from $Y_0^{(n)} = n$ which reaches 1 at the $\tau^{(n)}$ -th jump. The probability transitions of the Markov chain $Y^{(n)}$ are given by: for $b \geq 2$, $k \geq 1$ and $1 \leq l \leq b-1$,

$$(2) \quad p_{b,b-l} := \mathbb{P}(Y_k^{(n)} = b-l | Y_{k-1}^{(n)} = b) = \frac{\binom{b}{l+1} \lambda_{b,l+1}}{g_b},$$

and 1 is an absorbing state.

We introduce the discrete time process $X_k^{(n)} := Y_{k-1}^{(n)} - Y_k^{(n)}$, $k \geq 1$ with $X_0^{(n)} = 0$. This process counts the number of blocks we lose at the k -th jump. For $i \in \{1, \dots, n\}$, we define

$$T_i^{(n)} := \inf \left\{ t \mid \{i\} \notin \Pi_t^{(n)} \right\}$$

as the length of the i -th external branch and $T^{(n)}$ the length of a randomly chosen external branch. By exchangeability, $T_i^{(n)} \stackrel{(d)}{=} T^{(n)}$. We denote by $L_{ext}^{(n)} := \sum_{i=1}^n T_i^{(n)}$ the total external branch length of $\Pi^{(n)}$, and by $L^{(n)}$ the total branch length.

For several measures Λ , many asymptotic results on the external branches and their total external lengths of the Λ n -coalescent are already known.

- (1) If $\Lambda = \delta_0$, Dirac measure on 0, $\Pi^{(n)}$ is the Kingman n -coalescent. Then,
 - (a) $nT^{(n)}$ converges in distribution to T which is a random variable with density $f_T(x) = \frac{8}{(2+x)^3} \mathbf{1}_{x \geq 0}$ (See [7], [12], [26]).
 - (b) $L_{ext}^{(n)}$ converges in L^2 to 2 (see [21], [17]). A central limit theorem is also proved in [26].
- (2) If Λ is the uniform probability measure on $[0, 1]$, $\Pi^{(n)}$ is the Bolthausen-Sznitman n -coalescent. Then $(\log n)T^{(n)}$ converges in distribution to an exponential variable with parameter 1 (see [20], [40]). For moment results of $L_{ext}^{(n)}$, we refer to [14] and for central limit theorem, we refer to [29].
- (3) If $\nu_{-1} = \int_0^1 x^{-1} \Lambda(dx) < +\infty$, which includes the case of the $Beta(2 - \alpha, \alpha)$ -coalescent with $0 < \alpha < 1$, then
 - (a) $T^{(n)}$ converges in distribution to an exponential variable with parameter ν_{-1} (see [22, 36]).
 - (b) $L^{(n)}/n$ converges in distribution to a random variable L whose distribution coincides with that of $\int_0^{+\infty} e^{-X_t} dt$, where X_t is a certain subordinator (see page 1405 in [16] and [35]), and $L_{ext}^{(n)}/L^{(n)}$ converges in probability to 1 (see [36]).
- (4) If $\rho(t) = C_0 t^{-\alpha} + O(t^{-\alpha+\zeta})$, $C_0 > 0$, $\zeta > 1 - 1/\alpha$, $1 < \alpha < 2$, when $t \rightarrow 0$, which includes the $Beta(2 - \alpha, \alpha)$ -coalescents with $1 < \alpha < 2$ (with $C_0 = \frac{1}{\Gamma(\alpha+1)\Gamma(2-\alpha)}$), $n^{\alpha-1}T^{(n)}$ converges in distribution to T which is a random variable with density function (see [15])

$$(3) \quad f_T(x) = \frac{\alpha C_0 \Gamma(2 - \alpha)}{(\alpha - 1)} (1 + C_0 \Gamma(2 - \alpha) x)^{-\frac{\alpha}{\alpha-1}-1} \mathbf{1}_{x \geq 0}.$$

In the case of $Beta(2 - \alpha, \alpha)$ -coalescents with $1 < \alpha < 2$, we refer to [30, 28] for central limit theorems of $L_{ext}^{(n)}$ and $L^{(n)}$.

In this paper, we consider the processes which satisfy the following assumption:

$$(4) \quad \rho(t) = C_0 t^{-\alpha} + C_1 t^{-\alpha+\zeta} + o(t^{-\alpha+\zeta}), \quad C_0 > 0, C_1 \in \mathbb{R}, \zeta > 0, t \rightarrow 0.$$

The aim is to study the moments of the total external branch length $L_{ext}^{(n)}$ of these processes. We assume from now on that $1 < \alpha < 2$ and T is a random variable with density (3). Here is our main result.

For $s \in \mathbb{R}$, we define

$$(5) \quad \nu^{(s)}(dx) = (1-x)^s \nu(dx), \quad \text{and} \quad \omega^{(s)} = \sup\{u \geq 0; \forall 0 < y < 1, \int_y^1 (1-x)^{-u} \nu^{(s)}(dx) < +\infty\}.$$

We define in particular $\omega = \omega^{(0)}$ and notice that $\nu^{(0)} = \nu$. The quantity $\omega^{(s)}$ gives the information on the singularity of the measure $\nu^{(s)}$ near 1.

Theorem 1.1. *We assume that $\rho(t)$ satisfies (4) and that $\alpha - 1 < \zeta$, and $2(\alpha - 1) < \omega^{(1)}$.*

- (1) *The total external branch length $L_{ext}^{(n)}$ satisfies*

$$\lim_{n \rightarrow +\infty} n^{3\alpha-5} \mathbb{E}[(L_{ext}^{(n)} - n^{2-\alpha} \mathbb{E}[T])^2] = \frac{\Delta(\alpha)}{2},$$

where

$$\mathbb{E}[T] = \frac{\alpha - 1}{C_0 \Gamma(2 - \alpha)}, \quad \text{and} \quad \Delta(\alpha) = \frac{\int_0^1 ((1-x)^{2-\alpha} - 1)^2 \nu(dx)}{3 - \alpha} \left(\frac{\alpha - 1}{C_0 \Gamma(2 - \alpha)} \right)^3.$$

In particular, for the $Beta(2 - \alpha, \alpha)$ -coalescent

$$\lim_{n \rightarrow +\infty} n^{3\alpha-5} \mathbb{E}[(L_{ext}^{(n)} - n^{2-\alpha} \mathbb{E}[T])^2] = \frac{((\alpha-1)\Gamma(\alpha+1))^2 \Gamma(4-\alpha)}{2(3-\alpha)\Gamma(4-2\alpha)},$$

with $\mathbb{E}[T] = \alpha(\alpha-1)\Gamma(\alpha)$.

(2) As a consequence, $n^{\alpha-2} L_{ext}^{(n)} \xrightarrow{(L^2)} \mathbb{E}[T]$.

Remark 1.1. • For the second part of the theorem, the convergence in probability and almost surely can be deduced from [4], [5], [3] by Berestycki et al for a slightly different class of Λ coalescents.

- The first part of the theorem gives $n^{(5-3\alpha)/2}$ as the convergence speed for $L_{ext}^{(n)}$ tending to $n^{2-\alpha} \mathbb{E}[T]$ in the sense of second moment. But as shown in [30] for Beta($2-\alpha, \alpha$)-coalescents,

$$\frac{L_{ext}^{(n)} - n^{2-\alpha} \mathbb{E}[T]}{n^{1/\alpha+1-\alpha}} \xrightarrow{(d)} \frac{\alpha(2-\alpha)(\alpha-1)^{1/\alpha+1} \Gamma(\alpha)}{\Gamma(2-\alpha)^{1/\alpha}} \zeta,$$

where ζ is a stable random variable with parameter α . Our moment method fails to get the right speed of convergence in distribution.

To prove this result, the first idea is to write

$$(6) \quad \mathbb{E}[(L_{ext}^{(n)} - n^{2-\alpha} \mathbb{E}[T])^2] = n \text{Var}(T_1^{(n)}) + n(n-1) \text{Cov}(T_1^{(n)}, T_2^{(n)}) + (n \mathbb{E}[T_1^{(n)}] - n^{2-\alpha} \mathbb{E}[T])^2.$$

Hence we have to get results on the moments of the external branches. This is given by the next theorems. The first one gives the asymptotic behaviour for the covariance of two external branch lengths.

Theorem 1.2. *We assume that $\rho(t)$ satisfies (4) and that $\alpha-1 < \zeta$, and $2(\alpha-1) < \omega^{(1)}$. Then the asymptotic covariance of two external branch lengths is given by:*

$$\lim_{n \rightarrow +\infty} n^{3(\alpha-1)} \text{Cov}(T_1^{(n)}, T_2^{(n)}) = \Delta(\alpha).$$

Remark 1.2. We will prove a more general result: This theorem is the case (3) of Corollary 3.2. Notice that $\Delta(\alpha)$ is strictly positive implies that $\text{Cov}(T_1^{(n)}, T_2^{(n)})$ is of order $n^{3-3\alpha}$ and $T_1^{(n)}, T_2^{(n)}$ are positively correlated in the limit which is similar to Boltausen-Sznitman coalescent and opposite of Kingman coalescent (negatively correlated) (see [14]). To prove this theorem, we have to give the asymptotic behaviours of $\mathbb{E}[T_1^{(n)} T_2^{(n)}]$ (Theorem 3.1) and $\mathbb{E}[T_1^{(n)}]$ (Theorem 2.1). We also get from Theorem 2.1 that the third term in (6) satisfies

$$(7) \quad (n \mathbb{E}[T_1^{(n)}] - n^{2-\alpha} \mathbb{E}[T])^2 = O(n^{6-4\alpha}).$$

The second one gives the asymptotic behaviour of moments of one external branch length, hence we can estimate $n \text{Var}(T_1^{(n)})$.

Theorem 1.3. *We assume that $\rho(t)$ satisfies (4) and that $2(\alpha-1) < \omega^{(1)}$, and $\zeta > 1 - 1/\alpha$.*

(1) *If $0 \leq \beta < \frac{\alpha}{\alpha-1}$, $\beta(\alpha-1) < \omega^{(1)}$, then $\lim_{n \rightarrow +\infty} \mathbb{E}[(n^{\alpha-1} T_1^{(n)})^\beta] = \mathbb{E}[T^\beta]$.*

(2) *If $\beta \geq \frac{\alpha}{\alpha-1}$, then $\lim_{n \rightarrow +\infty} \mathbb{E}[(n^{\alpha-1} T_1^{(n)})^\beta] = +\infty$.*

Remark 1.3. Under the assumptions of Theorem 1.1, we can apply Theorem 1.3. Since $\frac{\alpha}{\alpha-1} > 2, \omega^{(1)} > 2(\alpha-1)$, Theorem 1.3 with $\beta = 1, 2$ leads to

$$\lim_{n \rightarrow +\infty} \text{Var}(n^{\alpha-1} T_1^{(n)}) = \text{Var}(T) < +\infty.$$

Hence $n \text{Var}(T_1^{(n)}) = O(n^{3-2\alpha})$. Recall that $(n \mathbb{E}[T_1^{(n)}] - n^{2-\alpha} \mathbb{E}[T])^2 = O(n^{6-4\alpha})$ (see (7)) and $n(n-1) \text{Cov}(T_1^{(n)}, T_2^{(n)}) = O(n^{5-3\alpha})$ (Theorem 1.2). Notice that we have $5-3\alpha > 6-4\alpha > 3-2\alpha$. Hence we deduce that $n(n-1) \text{Cov}(T_1^{(n)}, T_2^{(n)})$ is the dominant term in (6). Theorem 1.1 is then proved.

For the Beta($2-\alpha, \alpha$)-coalescent, Theorem 1.3 gives:

Corollary 1.4. *For Beta($2 - \alpha, \alpha$)-coalescent, we have*

- (1) *If $0 \leq \beta < \frac{\alpha}{\alpha-1}$, then $\lim_{n \rightarrow +\infty} \mathbb{E}[(n^{\alpha-1}T_1^{(n)})^\beta] = \mathbb{E}[T^\beta]$.*
- (2) *If $\beta \geq \frac{\alpha}{\alpha-1}$, then $\lim_{n \rightarrow +\infty} \mathbb{E}[(n^{\alpha-1}T_1^{(n)})^\beta] = +\infty$.*

1.3. Organization of this paper. We consider the Λ -coalescents which satisfy assumption (4). This includes Beta($2 - \alpha, \alpha$)-coalescents. In sections 2 and 3, we give estimates of $\mathbb{E}[T_1^{(n)}]$ and $\mathbb{E}[T_1^{(n)}T_2^{(n)}]$ respectively. Both $\mathbb{E}[T_1^{(n)}]$ and $\mathbb{E}[T_1^{(n)}T_2^{(n)}]$ satisfy the same kind of recurrence which allows to get their estimates and they lead to an estimate of $Cov(T_1^{(n)}, T_2^{(n)})$ in section 3. The main tool is Lemma 5.1 given in appendix A. In section 4, we deal with Theorem 1.3. Section 5 is the appendix where are given some proofs omitted before.

2. FIRST MOMENT OF $T_1^{(n)}$ BY RECURSIVE METHOD

2.1. The main result.

Theorem 2.1. *We assume that $\rho(t)$ satisfies (4).*

- (1) *If $0 < \zeta < \alpha - 1, \alpha - 1 + \zeta < \omega^{(1)}$, then*

$$\mathbb{E}[T_1^{(n)}] = \frac{\alpha - 1}{C_0\Gamma(2 - \alpha)}n^{1-\alpha} - \frac{C_1\Gamma(2 - \alpha + \zeta)(\alpha - 1)^2}{(C_0\Gamma(2 - \alpha))^2(\alpha - 1 - \zeta)}n^{1-\alpha-\zeta} + o(n^{1-\alpha-\zeta}).$$

- (2) *If $\zeta = \alpha - 1, 2(\alpha - 1) < \omega^{(1)}$, then*

$$\mathbb{E}[T_1^{(n)}] = \frac{\alpha - 1}{C_0\Gamma(2 - \alpha)}n^{1-\alpha} - \frac{(\alpha - 1)^2 C_1}{(C_0\Gamma(2 - \alpha))^2}n^{2(1-\alpha)} \ln n + o(n^{2(1-\alpha)} \ln n).$$

- (3) *If $\zeta > \alpha - 1, 2(\alpha - 1) < \omega^{(1)}$, then*

$$\begin{aligned} \mathbb{E}[T_1^{(n)}] &= \frac{\alpha - 1}{C_0\Gamma(2 - \alpha)}n^{1-\alpha} \\ &+ \frac{(\alpha - 1)^2}{C_0\Gamma(3 - \alpha)} \left(\frac{\int_0^1 ((1-x)^{1-\alpha} - 1 - (\alpha-1)x)\nu^{(1)}(dx)}{C_0\Gamma(2 - \alpha)} + \frac{(\alpha - 1)C_2^{(1)} - C_2}{C_0\Gamma(2 - \alpha)} \right) n^{2(1-\alpha)} \\ &+ o(n^{2(1-\alpha)}), \end{aligned}$$

$$\text{where } C_2 = \lim_{t \rightarrow 0} \int_t^1 \rho(r)dr - \frac{C_0 t^{1-\alpha}}{\alpha - 1}, C_2^{(1)} = \lim_{t \rightarrow 0} \int_t^1 \rho^{(1)}(r)dr - \frac{C_0 t^{1-\alpha}}{\alpha - 1}, \rho^{(1)}(t) = \int_t^1 \nu^{(1)}(dx).$$

Remark 2.1. For the Beta($2 - \alpha, \alpha$)-coalescent, we have $\omega^{(1)} = \alpha + 1, \zeta = 1$. Hence we are in case (3).

The idea is to use the recurrence satisfied by $T_1^{(n)}$ (see [14]):

$$(8) \quad \mathbb{E}[T_1^{(n)}] = \frac{1}{g_n} + \sum_{k=2}^{n-1} p_{n,k} \frac{k-1}{n} \mathbb{E}[T_1^{(k)}].$$

All the three cases in Theorem 2.1 will be proved in the same way. To give an idea of what should be done, we take case (1) as example. By defining $L = \frac{\alpha-1}{C_0\Gamma(2-\alpha)}, Q = \frac{C_1\Gamma(2-\alpha+\zeta)(\alpha-1)^2}{(C_0\Gamma(2-\alpha))^2(\alpha-1-\zeta)}$, we transform the recurrence (8) to

$$\begin{aligned} (9) \quad \left(\mathbb{E}[n^{\alpha-1}T_1^{(n)}] - L \right) n^\zeta + Q &= \left(\frac{n^{\alpha-1}}{g_n} - \left(1 - \sum_{k=1}^{n-1} p_{n,k} \frac{k-1}{n} \left(\frac{n}{k} \right)^{\alpha-1} \right) L \right) n^\zeta + Q \left(1 - \sum_{k=1}^{n-1} p_{n,k} \frac{k-1}{n} \left(\frac{n}{k} \right)^{\alpha-1+\zeta} \right) \\ &+ \sum_{k=2}^{n-1} \left(\frac{n}{k} \right)^{\alpha-1+\zeta} p_{n,k} \frac{k-1}{n} \left(k^\zeta (\mathbb{E}[k^{\alpha-1}T_1^{(k)}] - L) + Q \right). \end{aligned}$$

Hence we get a recurrence

$$(10) \quad a_n = b_n + \sum_{k=2}^{n-1} q_{n,k} a_k,$$

with

$$\begin{aligned} a_n &= \left(\mathbb{E}[n^{\alpha-1} T_1^{(n)}] - L \right) n^\zeta + Q, \\ b_n &= \left(\frac{n^{\alpha-1}}{g_n} - \left(1 - \sum_{k=2}^{n-1} p_{n,k} \frac{k-1}{n} \left(\frac{n}{k} \right)^{\alpha-1} \right) L \right) n^\zeta + Q \left(1 - \sum_{k=2}^{n-1} p_{n,k} \frac{k-1}{n} \left(\frac{n}{k} \right)^{\alpha-1+\zeta} \right), \\ q_{n,k} &= \left(\frac{n}{k} \right)^{\alpha-1+\zeta} p_{n,k} \frac{k-1}{n}. \end{aligned}$$

Then we should prove that $\lim_{n \rightarrow +\infty} a_n = 0$. It is natural to think about estimating b_n as n tends to infinity. To this aim, we should get asymptotics of g_n , $\frac{1}{g_n}$, and $\sum_{k=1}^{n-1} p_{n,k} \frac{(k-1)_l}{(n)_l} \left(\frac{n}{k} \right)^r$ with $r \geq 0$ and $l \in \mathbb{N}$, where $(n)_l$ is (the same for $(k-1)_l$):

$$(n)_l = \begin{cases} n(n-1)(n-2) \cdots (n-l+1) & \text{if } n \geq l \geq 1, \\ 0 & \text{if } l > n \geq 1. \end{cases}$$

2.2. Asymptotics of $1/g_n$. For any $c, d \in \mathbb{R}$, we have

$$(11) \quad \frac{\Gamma(n+c)}{\Gamma(n+d)} = n^{c-d} \left(1 + n^{-1} (c-d) \frac{c+d-1}{2} + O(n^{-2}) \right).$$

This is the straightforward consequence of Stirling's formula:

$$(12) \quad \Gamma(z) = \sqrt{2\pi} z^{z-1/2} e^{-z} \left(1 + \frac{1}{12z} + O\left(\frac{1}{z^2}\right) \right), \quad z > 0.$$

It is then easy to get: For real numbers a and $b > -1$,

$$(13) \quad \int_0^1 (1-t)^{n+a} t^b dx = \frac{\Gamma(n+a+1)\Gamma(b+1)}{\Gamma(n+a+b+2)} = \Gamma(b+1) n^{-1-b} \left(1 + n^{-1} (-1-b) \frac{b+2a+2}{2} + O(n^{-2}) \right).$$

Recall that $g_n = n(n-1) \int_0^1 t(1-t)^{n-2} \rho(t) dt$. Hence using (13), we get the following lemma.

Lemma 2.2. (1) *If $\rho(t)$ satisfies the condition:*

$$(14) \quad \rho(t) = C_0 t^{-\alpha} + o(t^{-\alpha}), \quad C_0 > 0.$$

Then

$$g_n = C_0 \Gamma(2-\alpha) n^\alpha + o(n^\alpha), \quad \text{and} \quad \frac{1}{g_n} = \frac{1}{C_0 \Gamma(2-\alpha)} n^{-\alpha} + o(n^{-\alpha}).$$

(2) *If $\rho(t)$ satisfies (4), then*

$$g_n = C_0 \Gamma(2-\alpha) n^\alpha - \frac{\alpha(\alpha-1)}{2} C_0 \Gamma(2-\alpha) n^{\alpha-1} + C_1 \Gamma(2-\alpha+\zeta) n^{\alpha-\zeta} + o(n^{\alpha-\zeta}) + O(1),$$

and

$$(15) \quad \frac{1}{g_n} = \begin{cases} \left(1 - \frac{C_1 \Gamma(2-\alpha+\zeta)}{C_0 \Gamma(2-\alpha)} n^{-\zeta} + o(n^{-\zeta}) \right) \frac{n^{-\alpha}}{C_0 \Gamma(2-\alpha)} & \text{if } 0 < \zeta < 1, \\ \left(1 + \left(\frac{\alpha(\alpha-1)}{2} - \frac{C_1 \Gamma(2-\alpha+\zeta)}{C_0 \Gamma(2-\alpha)} \right) n^{-1} + o(n^{-1}) \right) \frac{n^{-\alpha}}{C_0 \Gamma(2-\alpha)} & \text{if } \zeta = 1, \\ \left(1 + \frac{\alpha(\alpha-1)}{2} n^{-1} + o(n^{-1}) \right) \frac{n^{-\alpha}}{C_0 \Gamma(2-\alpha)} & \text{if } \zeta > 1. \end{cases}$$

2.3. Calculus of $\sum_{k=1}^{n-1} p_{n,k} \frac{(k-1)_l}{(n)_l} \left(\frac{n}{k}\right)^r$, $r \geq 0, l \in \mathbb{N}$. For any $s \in \mathbb{R}$, we denote by

$$g_n^{(s)} = \int_0^1 (1 - (1-x)^n - nx(1-x)^{n-1}) \nu^{(s)}(dx),$$

the collision rates of the Λ -coalescent associated with the measure $\nu^{(s)}$. We recall that $\nu^{(s)}$ is defined in (5). Notice that $g_n^{(0)} = g_n$.

Lemma 2.3. *Consider any Λ -coalescent process with measure ν . Let $l \in \{1, 2, \dots, n-2\}$ fixed. Then for any real function f :*

$$\sum_{k=1}^{n-1} p_{n,k} \frac{(k-1)_l}{(n)_l} f(k) = \mathbb{E} \left[\frac{(n-1-X_1^{(n)})_l}{(n)_l} \right] \mathbb{E}^{\nu^{(l)}} [f(n-X_1^{(n-l)})],$$

where $\mathbb{E}^{\nu^{(l)}}[\ast]$ means that the Λ -coalescent is associated with the measure $\nu^{(l)}$.

Proof. Recall the definitions of g_n and $p_{n,k}$ (see (1), (2)). We have

$$\begin{aligned} \sum_{k=1}^{n-1} p_{n,k} \frac{(k-1)_l}{(n)_l} &= \sum_{k=l+1}^{n-1} \frac{\int_0^1 \binom{n-l}{n-k+1} x^{n-k+1} (1-x)^{k-1} \nu(dx)}{g_n} \\ &= \sum_{k=l+1}^{n-1} \frac{\int_0^1 \binom{n-l}{n-k+1} x^{n-k+1} (1-x)^{k-1-l} \nu^{(l)}(dx)}{g_n} \\ (16) \quad &= \sum_{k=1}^{n-1-l} \frac{\int_0^1 \binom{n-l}{n-k-l+1} x^{n-k-l+1} (1-x)^{k-1} \nu^{(l)}(dx)}{g_n} = \frac{g_{n-l}^{(l)}}{g_n}. \end{aligned}$$

Then,

$$\begin{aligned} \sum_{k=1}^{n-1} p_{n,k} \frac{(k-1)_l}{(n)_l} f(k) &= \left(\sum_{k=1}^{n-1} p_{n,k} \frac{(k-1)_l}{(n)_l} \right) \frac{\sum_{k=1}^{n-1} p_{n,k} \frac{(k-1)_l}{(n)_l} f(k)}{\sum_{k=1}^{n-1} p_{n,k} \frac{(k-1)_l}{(n)_l}} \\ &= \mathbb{E} \left[\frac{(n-1-X_1^{(n)})_l}{(n)_l} \right] \frac{\sum_{k=l+1}^{n-1} \int_0^1 \binom{n-l}{n-k+1} x^{n-k+1} (1-x)^{k-1-l} f(k) \nu^{(l)}(dx)}{g_{n-l}^{(l)}} \\ &= \mathbb{E} \left[\frac{(n-1-X_1^{(n)})_l}{(n)_l} \right] \frac{\sum_{k=1}^{n-1-l} \int_0^1 \binom{n-l}{n-k-l+1} x^{n-k-l+1} (1-x)^{k-1} f(k+l) \nu^{(l)}(dx)}{g_{n-l}^{(l)}} \\ &= \mathbb{E} \left[\frac{(n-1-X_1^{(n)})_l}{(n)_l} \right] \mathbb{E}^{\nu^{(l)}} [f(Y_1^{(n-l)} + l)] = \mathbb{E} \left[\frac{(n-1-X_1^{(n)})_l}{(n)_l} \right] \mathbb{E}^{\nu^{(l)}} [f(n-X_1^{(n-l)})]. \end{aligned}$$

This achieves the proof of the lemma. \square

In consequence,

$$(17) \quad \sum_{k=1}^{n-1} p_{n,k} \frac{(k-1)_l}{(n)_l} \left(\frac{n}{k}\right)^r = \mathbb{E} \left[\frac{(n-1-X_1^{(n)})_l}{(n)_l} \right] \mathbb{E}^{\nu^{(l)}} \left[\left(\frac{n}{n-X_1^{(n-l)}} \right)^r \right].$$

We have to study $\mathbb{E} \left[\frac{(n-1-X_1^{(n)})_l}{(n)_l} \right]$ and $\mathbb{E}^{\nu^{(l)}} \left[\left(\frac{n}{n-X_1^{(n-l)}} \right)^r \right]$. The latter is very close to Proposition 5.4 in appendix B. The following lemma studies the former.

Lemma 2.4. *We assume that $\rho(t)$ satisfies (4). Let $l \in \{1, 2, \dots, n-2\}$ fixed.*

(1) If $0 < \zeta < \alpha - 1$, then

$$\mathbb{E}\left[\frac{(n-1-X_1^{(n)})_l}{(n)_l}\right] = 1 - \frac{l\alpha}{n(\alpha-1)} - \frac{lC_1\Gamma(2-\alpha+\zeta)\zeta}{C_0(\alpha-1)\Gamma(2-\alpha)(\alpha-1-\zeta)}n^{-\zeta-1} + o(n^{-\zeta-1}).$$

(2) If $\zeta = \alpha - 1$, then

$$\mathbb{E}\left[\frac{(n-1-X_1^{(n)})_l}{(n)_l}\right] = 1 - \frac{l\alpha}{n(\alpha-1)} - \frac{lC_1}{C_0\Gamma(2-\alpha)}n^{-\alpha}\ln n + o(n^{-\alpha}\ln n).$$

(3) If $\zeta > \alpha - 1$, then

$$\mathbb{E}\left[\frac{(n-1-X_1^{(n)})_l}{(n)_l}\right] = 1 - \frac{l\alpha}{n(\alpha-1)} + \left(\sum_{j=2}^l \binom{l}{j} (-1)^j \frac{\int_0^1 jx^{j-1}\rho(x)dx}{C_0\Gamma(2-\alpha)} - \frac{C_2l}{C_0\Gamma(2-\alpha)}\right)n^{-\alpha} + o(n^{-\alpha}),$$

where C_2 is the same as in Theorem 2.1.

Proof. We have

$$\mathbb{E}\left[\frac{(n-1-X_1^{(n)})_l}{(n)_l}\right] = \mathbb{E}\left[1 - \sum_{i=0}^{l-1} \frac{X_1^{(n)} + 1}{n-i} + \sum_{j=2}^l \sum_{i_1, i_2, \dots, i_j \text{ all different}} (-1)^j \frac{(X_1^{(n)} + 1)^j}{(n-i_1)(n-i_2)\cdots(n-i_j)}\right].$$

For $\mathbb{E}[\sum_{i=0}^{l-1} \frac{X_1^{(n)} + 1}{n-i}]$, we use Lemma 5.2 in appendix B. While using Lemme 5.3, we get

$$\mathbb{E}\left[\sum_{j=2}^l \sum_{i_1, i_2, \dots, i_j \text{ all different}} (-1)^j \frac{(X_1^{(n)} + 1)^j}{(n-i_1)(n-i_2)\cdots(n-i_j)}\right] = n^{-\alpha} \sum_{j=2}^l \binom{l}{j} (-1)^j \frac{\int_0^1 jx^{j-1}\rho(x)dx}{C_0\Gamma(2-\alpha)} + O(n^{-\min\{1+\alpha, j\}}). \quad (18)$$

Then we conclude. □

Now we can finally give the estimate of $\sum_{k=1}^{n-1} p_{n,k} \frac{(k-1)_l}{(n)_l} \left(\frac{n}{k}\right)^r$.

Proposition 2.5. *We assume that $\rho(t)$ satisfies (4). Let $l \in \{1, 2, \dots, n-2\}$ fixed and $r \in [0, \omega^{(l)}]$.*

(1) If $0 < \zeta < \alpha - 1$, then

$$\sum_{k=1}^{n-1} p_{n,k} \frac{(k-1)_l}{(n)_l} \left(\frac{n}{k}\right)^r = 1 + \frac{r-l\alpha}{n(\alpha-1)} + \frac{(r-l)C_1\Gamma(2-\alpha+\zeta)\zeta}{C_0(\alpha-1)\Gamma(2-\alpha)(\alpha-1-\zeta)}n^{-\zeta-1} + o(n^{-\zeta-1}).$$

(2) If $\zeta = \alpha - 1$, then

$$\sum_{k=1}^{n-1} p_{n,k} \frac{(k-1)_l}{(n)_l} \left(\frac{n}{k}\right)^r = 1 + \frac{r-l\alpha}{n(\alpha-1)} + \frac{(r-l)C_1}{C_0\Gamma(2-\alpha)}n^{-\alpha}\ln n + o(n^{-\alpha}\ln n).$$

(3) If $\zeta > \alpha - 1$, then

$$\begin{aligned} \sum_{k=1}^{n-1} p_{n,k} \frac{(k-1)_l}{(n)_l} \left(\frac{n}{k}\right)^r &= 1 + \frac{(r-l\alpha)}{n(\alpha-1)} + \left(\frac{\int_0^1 ((1-x)^{-r} - 1 - rx)\nu^{(l)}(dx)}{C_0\Gamma(2-\alpha)} \right. \\ &\quad \left. + \sum_{j=2}^l \binom{l}{j} (-1)^j \frac{\int_0^1 jx^{j-1}\rho(x)dx}{C_0\Gamma(2-\alpha)} + \frac{rC_2^{(l)} - lC_2}{C_0\Gamma(2-\alpha)}\right)n^{-\alpha} + o(n^{-\alpha}), \end{aligned}$$

where C_2 is the same as in Theorem 2.1 and $C_2^{(l)} = \lim_{t \rightarrow 0} \int_t^1 \rho^{(l)}(r) dr - \frac{C_0 t^{1-\alpha}}{\alpha-1}$.

Proof. The proof is a consequence of (17) and Remark 5.2. \square

Remark 2.2. The Remark 5.3 following Proposition 5.4 implies that if $r \geq \omega^{(r)}$, this proposition is not valid at least in case (3). Since this proposition is critical for the proof of Theorem 2.1 (this will be shown in subsection 2.4), the assumptions about α and $\omega^{(1)}$ made in Theorem 2.1 are necessary.

2.4. Proof of Theorem 2.1.

- (1) Recall the transformation (9) and the associated recurrence (10). The aim is to prove that $\lim_{n \rightarrow +\infty} a_n = 0$ for a_n in (10). Under the assumptions $0 < \zeta < \alpha - 1$, and $\alpha - 1 + \zeta < \omega^{(1)}$, using Proposition 2.5, we get

$$1 - \sum_{k=1}^{n-1} p_{n,k} \frac{k-1}{n} \left(\frac{n}{k}\right)^{\alpha-1} = \frac{1}{n(\alpha-1)} - \frac{(\alpha-2)C_1\Gamma(2-\alpha+\zeta)\zeta}{C_0(\alpha-1)\Gamma(2-\alpha)(\alpha-1-\zeta)} n^{-\zeta-1} + o(n^{-\zeta-1}),$$

$$1 - \sum_{k=1}^{n-1} p_{n,k} \frac{k-1}{n} \left(\frac{n}{k}\right)^{\alpha-1+\zeta} = \frac{1-\zeta}{n(\alpha-1)} - \frac{(\alpha-2+\zeta)C_1\Gamma(2-\alpha+\zeta)\zeta}{C_0(\alpha-1)\Gamma(2-\alpha)(\alpha-1-\zeta)} n^{-\zeta-1} + o(n^{-\zeta-1}),$$

and then $b_n = o(n^{-1})$ by (15). Let $\varepsilon > 0$ such that $\alpha - 1 + \zeta + \varepsilon < \omega^{(1)}$. Then we have $1 - \sum_{k=1}^{n-1} p_{n,k} \frac{k-1}{n} \left(\frac{n}{k}\right)^{\alpha-1+\zeta+\varepsilon} = O(n^{-1}) > 0$. Hence the recurrence (10) satisfies the assumptions of Lemma 5.1 which leads to $\lim_{n \rightarrow +\infty} a_n = 0$. Then we can conclude.

- (2) Similarly, we transform recurrence (8) to

$$\begin{aligned} & \frac{n^{\alpha-1}}{\ln n} \left(\mathbb{E}[n^{\alpha-1} T_1^{(n)}] - L \right) + Q \\ &= \frac{n^{\alpha-1}}{\ln n} \left(\frac{n^{\alpha-1}}{g_n} - \left(1 - \sum_{k=1}^{n-1} p_{n,k} \frac{k-1}{n} \left(\frac{n}{k}\right)^{\alpha-1} \right) L \right) + Q \left(1 - \sum_{k=1}^{n-1} p_{n,k} \frac{k-1}{n} \left(\frac{n}{k}\right)^{2(\alpha-1)} \frac{\ln k}{\ln n} \right) \\ &+ \sum_{k=2}^{n-1} \frac{\ln k}{\ln n} p_{n,k} \frac{k-1}{n} \left(\frac{n}{k}\right)^{2(\alpha-1)} \left(\frac{k^{\alpha-1}}{\ln k} (\mathbb{E}[k^{\alpha-1} T_1^{(k)}] - L) + Q \right), \end{aligned}$$

where $L = \frac{\alpha-1}{C_0\Gamma(2-\alpha)}$, $Q = \frac{(\alpha-1)^2 C_1}{(C_0\Gamma(2-\alpha))^2}$. Again, we denote by

$$\begin{aligned} a_n &= \frac{n^{\alpha-1}}{\ln n} \left(\mathbb{E}[n^{\alpha-1} T_1^{(n)}] - L \right) + Q, \\ b_n &= \frac{n^{\alpha-1}}{\ln n} \left(\frac{n^{\alpha-1}}{g_n} - \left(1 - \sum_{k=1}^{n-1} p_{n,k} \frac{k-1}{n} \left(\frac{n}{k}\right)^{\alpha-1} \right) L \right) + Q \left(1 - \sum_{k=1}^{n-1} p_{n,k} \frac{k-1}{n} \left(\frac{n}{k}\right)^{2(\alpha-1)} \frac{\ln k}{\ln n} \right), \\ q_{n,k} &= \left(\frac{n}{k}\right)^{2(\alpha-1)} \frac{\ln k}{\ln n} p_{n,k} \frac{k-1}{n}, \end{aligned}$$

so that $a_n = b_n + \sum_{k=2}^{n-1} q_{n,k} a_k$.

Here the only thing that we do not know is the estimate of $\sum_{k=1}^{n-1} p_{n,k} \frac{k-1}{n} \left(\frac{n}{k}\right)^{2(\alpha-1)} \frac{\ln k}{\ln n}$. Using the same idea as in Proposition 2.5, we prove that : If $\rho(t)$ satisfies condition (4) with $\zeta = \alpha - 1$, for l fixed in $\{1, 2, \dots, n-2\}$ and $r \in [0, \omega^{(l)}]$,

$$(19) \quad \sum_{k=1}^{n-1} p_{n,k} \frac{(k-1)^l}{(n)^l} \left(\frac{n}{k}\right)^r \left(\frac{\ln k}{\ln n}\right)^\beta = 1 + \frac{r-l\alpha}{n(\alpha-1)} - \frac{\beta}{(\alpha-1)n \ln n} + \frac{(r-l)C_1}{C_0\Gamma(2-\alpha)} n^{-\alpha} \ln n + o(n^{-\alpha} \ln n).$$

Every step is identical to the proof of the first case to get $\lim_{n \rightarrow +\infty} a_n = 0$. We deduce that

$$\mathbb{E}[T_1^{(n)}] = Ln^{1-\alpha} - Qn^{2(1-\alpha)} \ln n + o(n^{2(1-\alpha)} \ln n).$$

- (3) This case is similar to the first case, so we skip it for simplicity.

3. ESTIMATE OF $\text{COV}(T_1^{(n)}, T_2^{(n)})$

Using Theorem 1.1 of [14], we have

$$(20) \quad \mathbb{E}[T_1^{(n)} T_2^{(n)}] = \frac{2\mathbb{E}[T_1^{(n)}]}{g_n} + \sum_{k=1}^{n-1} p_{n,k} \frac{(k-1)_2}{(n)_2} \mathbb{E}[T_1^{(k)} T_2^{(k)}].$$

The estimate of $\mathbb{E}[T_1^{(n)} T_2^{(n)}]$ can be obtained in the same way as in the proof of Theorem 2.1. At first, we give an estimate of $\frac{2\mathbb{E}[T_1^{(n)}]}{g_n}$ which is the consequence of (15) and Theorem 2.1. We assume that $\rho(t)$ satisfies (4).

(1) If $0 < \zeta < \alpha - 1, \alpha - 1 + \zeta < \omega^{(1)}$, then

$$\frac{2\mathbb{E}[T_1^{(n)}]}{g_n} = \frac{2n^{1-2\alpha}}{C_0\Gamma(2-\alpha)} \left(\frac{\alpha-1}{C_0\Gamma(2-\alpha)} - \frac{C_1\Gamma(2-\alpha+\zeta)(\alpha-1)(2\alpha-2-\zeta)}{(C_0\Gamma(2-\alpha))^2(\alpha-1-\zeta)} n^{-\zeta} + o(n^{-\zeta}) \right).$$

(2) If $\zeta = \alpha - 1, 2(\alpha - 1) < \omega^{(1)}$, then

$$\frac{2\mathbb{E}[T_1^{(n)}]}{g_n} = \frac{2n^{1-2\alpha}}{C_0\Gamma(2-\alpha)} \left(\frac{\alpha-1}{C_0\Gamma(2-\alpha)} - \frac{C_1(\alpha-1)^2}{(C_0\Gamma(2-\alpha))^2} n^{1-\alpha} \ln n + o(n^{1-\alpha} \ln n) \right).$$

(3) If $\zeta > \alpha - 1, 2(\alpha - 1) < \omega^{(1)}$, then

$$\begin{aligned} \frac{2\mathbb{E}[T_1^{(n)}]}{g_n} &= \frac{2n^{1-2\alpha}}{C_0\Gamma(2-\alpha)} \left(\frac{\alpha-1}{C_0\Gamma(2-\alpha)} \right. \\ &\quad \left. + \frac{(\alpha-1)^2}{C_0\Gamma(3-\alpha)} \left(\frac{\int_0^1 ((1-x)^{1-\alpha} - 1 - (\alpha-1)x)\nu^{(1)}(dx)}{C_0\Gamma(2-\alpha)} + \frac{(\alpha-1)C_2^{(1)} - C_2}{C_0\Gamma(2-\alpha)} \right) n^{1-\alpha} + o(n^{1-\alpha}) \right). \end{aligned}$$

We give the result and leave the proof to the reader.

Theorem 3.1. *We assume that $\rho(t)$ satisfies (4).*

(1) *If $0 < \zeta < \alpha - 1, (\alpha - 1) + \zeta < \omega^{(1)}$, then*

$$\mathbb{E}[T_1^{(n)} T_2^{(n)}] = \left(\frac{\alpha-1}{C_0\Gamma(2-\alpha)} \right)^2 n^{2(1-\alpha)} - \frac{2C_1\Gamma(2-\alpha+\zeta)}{\alpha-1-\zeta} \left(\frac{\alpha-1}{C_0\Gamma(2-\alpha)} \right)^3 n^{2(1-\alpha)-\zeta} + o(n^{2(1-\alpha)-\zeta}).$$

(2) *If $\zeta = \alpha - 1, 2(\alpha - 1) < \omega^{(1)}$, then*

$$\mathbb{E}[T_1^{(n)} T_2^{(n)}] = \left(\frac{\alpha-1}{C_0\Gamma(2-\alpha)} \right)^2 n^{2(1-\alpha)} - \frac{2C_1(\alpha-1)^3}{(C_0\Gamma(2-\alpha))^3} n^{3(1-\alpha)} \ln n + o(n^{3(1-\alpha)} \ln n).$$

(3) *If $\zeta > \alpha - 1, 2(\alpha - 1) < \omega^{(1)}$, then*

$$\begin{aligned} \mathbb{E}[T_1^{(n)} T_2^{(n)}] &= \left(\frac{\alpha-1}{C_0\Gamma(2-\alpha)} \right)^2 n^{2(1-\alpha)} \\ &\quad + \frac{\alpha-1}{3-\alpha} \left(\frac{\alpha-1}{C_0\Gamma(2-\alpha)} \right)^2 \left(B + \frac{2(\alpha-1)C_2^{(2)} + \int_0^1 2t\rho(t)dt - 2C_2}{C_0\Gamma(2-\alpha)} + \frac{2}{2-\alpha} \left(A + \frac{(\alpha-1)C_2^{(1)} - C_2}{C_0\Gamma(2-\alpha)} \right) \right) n^{3(1-\alpha)} \\ &\quad + o(n^{3(1-\alpha)}), \end{aligned}$$

where C_2 and $C_2^{(l)}$ are the same as in Proposition 2.5, and

$$A = \frac{\int_0^1 ((1-x)^{1-\alpha} - 1 - (\alpha-1)x) \nu^{(1)}(dx)}{C_0 \Gamma(2-\alpha)}, B = \frac{\int_0^1 ((1-x)^{2(1-\alpha)} - 1 - 2(\alpha-1)x) \nu^{(2)}(dx)}{C_0 \Gamma(2-\alpha)}.$$

Now combining Theorem 2.1 and 3.1, we can get the estimate of $\text{Cov}(T_1^{(n)}, T_2^{(n)})$.

Corollary 3.2. *We assume that $\rho(t)$ satisfies (4).*

(1) *If $0 < \zeta < \alpha - 1, \alpha - 1 + \zeta < \omega^{(1)}$, then*

$$\text{Cov}(T_1^{(n)}, T_2^{(n)}) = o(n^{2(1-\alpha)-\zeta}).$$

(2) *If $\zeta = \alpha - 1, 2(\alpha - 1) < \omega^{(1)}$, then*

$$\text{Cov}(T_1^{(n)}, T_2^{(n)}) = o(n^{3(1-\alpha)} \ln n).$$

(3) *If $\zeta > \alpha - 1, 2(\alpha - 1) < \omega^{(1)}$, then*

$$\text{Cov}(T_1^{(n)}, T_2^{(n)}) = \Delta(\alpha) n^{3(1-\alpha)} + o(n^{3(1-\alpha)}),$$

$$\text{where } \Delta(\alpha) = \frac{\int_0^1 ((1-x)^{2-\alpha} - 1)^2 \nu(dx)}{3-\alpha} \left(\frac{\alpha-1}{C_0 \Gamma(2-\alpha)} \right)^3.$$

Proof. We skip the easy cases (1) and (2). For case (3), we get from Theorem 2.1 and 3.1 that

$$\text{Cov}(T_1^{(n)}, T_2^{(n)}) = \frac{n^{3(1-\alpha)}}{3-\alpha} \frac{(\alpha-1)^3}{(C_0 \Gamma(2-\alpha))^2} (B - 2A + \frac{2(\alpha-1)(C_2^{(2)} - C_2^{(1)}) + \int_0^1 2t\rho(t)dt}{C_0 \Gamma(2-\alpha)}) + o(n^{3(1-\alpha)}).$$

Hence

$$(21) \quad \Delta(\alpha) = \frac{(\alpha-1)^3}{(3-\alpha)(C_0 \Gamma(2-\alpha))^2} (B - 2A + \frac{2(\alpha-1)(C_2^{(2)} - C_2^{(1)}) + \int_0^1 2t\rho(t)dt}{C_0 \Gamma(2-\alpha)}).$$

Notice that

$$B - 2A = \frac{\int_0^1 ((1-x)^{2(2-\alpha)} - 2(1-x)^{2-\alpha} + 1 - x^2 + 2(\alpha-1)x^2(1-x)) \nu(dx)}{C_0 \Gamma(2-\alpha)}.$$

By definition,

$$C_2^{(2)} - C_2^{(1)} = \lim_{t \rightarrow +\infty} \int_t^1 (\rho^{(2)}(x) - \rho^{(1)}(x)) dx = \lim_{t \rightarrow 0} \int_t^1 x(\nu^{(2)}(dx) - \nu^{(1)}(dx)) = \int_0^1 -x^2(1-x) \nu(dx),$$

and $\int_0^1 2t\rho(t)dt = \int_0^1 x^2 \nu(dx)$. Then we get the result. \square

3.1. Beta($2-\alpha, \alpha$). For the Beta($2-\alpha, \alpha$)-coalescent, we have $\zeta = 1, \omega^{(1)} = 1+\alpha$, so $\zeta > \alpha-1, \omega^{(1)} > 2(\alpha-1)$. Hence this process is in the case (3) of Corollary 3.2. We can compute explicitly the constants $C_2, C_2^{(1)}, C_2^{(2)}, A, B$ in Theorem 3.1:

$$(1) \quad C_2 = \frac{1}{1-\alpha}, C_2^{(1)} = \frac{\alpha}{1-\alpha}, C_2^{(2)} = \frac{\alpha^2 + \alpha}{2(1-\alpha)}.$$

$$(2) \quad A = \alpha(\alpha^2 - \alpha - 1)\Gamma(\alpha - 1), B = \frac{1}{(\alpha-1)} \left(\frac{\Gamma(4-\alpha)}{\Gamma(4-2\alpha)} + (\alpha^2 - \alpha - 1)\Gamma(\alpha + 2) \right).$$

$$(3) \quad \Delta(\alpha) = \frac{((\alpha-1)\Gamma(\alpha+1))^2 \Gamma(4-\alpha)}{(3-\alpha)\Gamma(4-2\alpha)}.$$

Proof. (1) For Beta($2-\alpha, \alpha$)-coalescent, we have for $C_0 = \frac{1}{\alpha\Gamma(\alpha)\Gamma(2-\alpha)}$ and for $l = 0, 1, 2$,

$$\nu^{(l)}(dx) = \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)} x^{-1-\alpha} (1-x)^{\alpha-1+l} dx, \quad \text{and} \quad \rho^{(l)}(t) = \frac{1}{\alpha\Gamma(\alpha)\Gamma(2-\alpha)} t^{-\alpha} + O(t^{1-\alpha}).$$

Using integration by parts,

$$\begin{aligned}
\int_t^1 \rho^{(l)}(x) dx &= -t\rho^{(l)}(t) + \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)} \int_t^1 x^{-\alpha}(1-x)^{\alpha-1+l} dx \\
&= -\frac{1}{\alpha\Gamma(\alpha)\Gamma(2-\alpha)} t^{1-\alpha} + \frac{1}{(\alpha-1)\Gamma(\alpha)\Gamma(2-\alpha)} t^{1-\alpha} \\
&\quad + \frac{\alpha+l-1}{(1-\alpha)\Gamma(\alpha)\Gamma(2-\alpha)} \int_t^1 x^{1-\alpha}(1-x)^{\alpha+l-2} dx + O(t^{2-\alpha}) \\
&= \frac{C_0}{\alpha-1} t^{1-\alpha} + \frac{\Gamma(\alpha+l)}{\Gamma(l+1)\Gamma(\alpha)(1-\alpha)} + O(t^{2-\alpha}).
\end{aligned}$$

The choices $l = 0, 1, 2$, give (1).

(2) Recall that

$$A = \frac{\int_0^1 ((1-x)^{1-\alpha} - 1 - (\alpha-1)x) \nu^{(1)}(dx)}{C_0\Gamma(2-\alpha)} = \frac{\alpha}{\Gamma(2-\alpha)} \int_0^1 ((1-x)^{1-\alpha} - 1 - (\alpha-1)x) x^{-1-\alpha} (1-x)^\alpha dx.$$

Using integration by parts two times,

$$\begin{aligned}
A &= \frac{\alpha}{\Gamma(2-\alpha)} \frac{1}{\alpha(\alpha-1)} \int_0^1 x^{1-\alpha} (-\alpha(\alpha-1)(1-x)^{\alpha-2} + 2\alpha(\alpha-1)(1-x)^{\alpha-1} - \alpha(\alpha-1)^2 x(1-x)^{\alpha-2}) dx \\
&= \frac{1}{\Gamma(2-\alpha)(\alpha-1)} (-\Gamma(\alpha+1)\Gamma(2-\alpha) + 2(\alpha-1)\Gamma(\alpha+1)\Gamma(2-\alpha) - (\alpha-1)\Gamma(3-\alpha)\Gamma(\alpha+1)) \\
&= \alpha(\alpha^2 - \alpha - 1)\Gamma(\alpha-1).
\end{aligned}$$

B can be computed directly using integration by parts two times.

(3) Notice that $\Delta(\alpha)$ has the expression (21). Hence it can be calculated since we know $A, B, C_2^{(1)}$, and $C_2^{(2)}$. We omit the very detailed calculus. \square

4. PROOF OF THEOREM 1.3

Notice that $n^{\alpha-1}T_1^{(n)} \xrightarrow{(d)} T$ and if $\beta \geq \frac{\alpha}{\alpha-1}$, one gets $\mathbb{E}[T^\beta] = +\infty$, hence $\mathbb{E}[(n^{\alpha-1}T_1^{(n)})^\beta]$ converges to $+\infty$ (see Lemma 4.11 of [27]). If $0 \leq \beta_1 < \beta_2 < \frac{\alpha}{\alpha-1}$ and $(\mathbb{E}[(n^{\alpha-1}T_1^{(n)})^{\beta_2}], n \geq 2)$ is bounded. Then $((n^{\alpha-1}T_1^{(n)})^{\beta_1}, n \geq 2)$ is uniformly integrable (see Lemma 4.11 of [27] and Problem 14 in section 8.3 [11]). Then we need only to prove that for $\beta \in [2, \frac{\alpha}{\alpha-1})$, $(\mathbb{E}[(n^{\alpha-1}T_1^{(n)})^\beta], n \geq 2)$ is bounded.

We will prove by induction on n that there exists a constant $C > 0$ such that for all $n \geq 2$, $(\mathbb{E}[n^{\alpha-1}T_1^{(n)}])^\beta \leq C$. We first assume that, for all $2 \leq k \leq n-1$, $(\mathbb{E}[k^{\alpha-1}T_1^{(k)}])^\beta \leq C$ and then will prove that (if C is large enough) $(\mathbb{E}[n^{\alpha-1}T_1^{(n)}])^\beta \leq C$.

Writing the decomposition of $T_1^{(n)}$ at the first coalescence, we have

$$T_1^{(n)} = \frac{e_0}{g_n} + \sum_{k=2}^{n-1} \mathbf{1}_{\{H_{n,k}\}} \bar{T}_1^{(k)},$$

where:

- $H_{n,k}$ is the event: {From n individuals, we have k individuals after the first coalescence, and individual 1 is not involved in this collision}, $2 \leq k \leq n-1$;
- e_0 is a unit exponential random variable, $\bar{T}_1^{(k)} \stackrel{(d)}{=} T_1^{(k)}$, and all these random variables $e_0, \bar{T}_1^{(k)}, \mathbf{1}_{\{H_{n,k}\}}$ are independent. One notices that $\mathbb{P}(H_{n,k}) = p_{n,k} \frac{k-1}{n}$, compared to (8).

Thanks to Lemma 5.8 in Appendix C, we have the following inequality.

$$(22) \quad \mathbb{E}[(T_1^{(n)})^\beta] = \mathbb{E}\left[\left(\frac{e_0}{g_n} + \sum_{k=2}^{n-1} \mathbf{1}_{\{H_{n,k}\}} \bar{T}_1^{(k)}\right)^\beta\right] \leq I_{n,1} + I_{n,2} + I_{n,3} + I_{n,4}$$

where

$$I_{n,1} = \mathbb{E}\left[\left(\frac{e_0}{g_n}\right)^\beta\right], \quad I_{n,2} = \mathbb{E}\left[\left(\sum_{k=2}^{n-1} \mathbf{1}_{\{H_{n,k}\}} \bar{T}_1^{(k)}\right)^\beta\right],$$

$$I_{n,3} = \mathbb{E}\left[\beta 2^{\beta-1} \frac{e_0}{g_n} \left(\sum_{k=2}^{n-1} \mathbf{1}_{\{H_{n,k}\}} \bar{T}_1^{(k)}\right)^{\beta-1}\right] \text{ and } I_{n,4} = \mathbb{E}\left[\beta 2^{\beta-1} \left(\frac{e_0}{g_n}\right)^{\beta-1} \sum_{k=2}^{n-1} \mathbf{1}_{\{H_{n,k}\}} \bar{T}_1^{(k)}\right].$$

We first bound $I_{n,1}$. Recall that $g_n \sim C_0 \Gamma(2 - \alpha) n^\alpha$, hence there exists a constant $K_1 > 0$ such that for $\beta \geq 2 > 1$ and $n \geq 2$,

$$(23) \quad n^{(\alpha-1)\beta} I_{n,1} \leq \frac{K_1}{n}.$$

We now consider $I_{n,2}$. Notice that $(\alpha - 1)\beta < \omega^{(1)}$. Hence, using Proposition 2.5, we have

$$(24) \quad n^{(\alpha-1)\beta} I_{n,2} = n^{-(\alpha-1)\beta} \sum_{k=2}^{n-1} p_{n,k} \frac{k-1}{n} \left(\frac{n}{k}\right)^{(\alpha-1)\beta} \mathbb{E}[(k^{\alpha-1} T_1^{(k)})^\beta]$$

$$(25) \quad \leq C \sum_{k=2}^{n-1} p_{n,k} \frac{k-1}{n} \left(\frac{n}{k}\right)^{(\alpha-1)\beta}$$

$$(26) \quad = C \left(1 - \frac{\alpha - (\alpha - 1)\beta}{n(\alpha - 1)} + o(n^{-1})\right) \leq C \left(1 - \frac{\alpha - (\alpha - 1)\beta}{2n(\alpha - 1)}\right),$$

for $n \geq N$, where N is a fixed positive integer.

We now proceed to $I_{n,3}$. Notice that for $2 \leq k \leq n-1$,

$$\mathbb{E}[(k^{\alpha-1} T_1^{(k)})^{\beta-1}] \leq (\mathbb{E}[(k^{\alpha-1} T_1^{(k)})^\beta])^{\frac{\beta-1}{\beta}} \leq C^{\frac{\beta-1}{\beta}}.$$

Hence we have

$$(27) \quad \begin{aligned} n^{(\alpha-1)\beta} I_{n,3} &= n^{(\alpha-1)\beta} \mathbb{E}\left[\beta 2^{\beta-1} \frac{e_0}{g_n} \sum_{k=2}^{n-1} \mathbf{1}_{\{H_{n,k}\}} (\bar{T}_1^{(k)})^{\beta-1}\right] \\ &\leq C^{\frac{\beta-1}{\beta}} \beta 2^{\beta-1} n^{\alpha-1} g_n^{-1} \sum_{k=2}^{n-1} p_{n,k} \frac{k-1}{n} \left(\frac{n}{k}\right)^{(\alpha-1)(\beta-1)} \\ &= C^{\frac{\beta-1}{\beta}} n^{\alpha-1} \beta 2^{\beta-1} g_n^{-1} \left(1 - \frac{\alpha - (\alpha - 1)(\beta - 1)}{n(\alpha - 1)} + o(n^{-1})\right) \\ &\leq \frac{C^{\frac{\beta-1}{\beta}} K_2}{n}, \end{aligned}$$

where K_2 is a positive constant. In the second equality, we have used Proposition 2.5.

While for any $n \geq 2$,

$$(28) \quad \begin{aligned} n^{(\alpha-1)\beta} I_{n,4} &= n^{(\alpha-1)\beta} \mathbb{E}\left[\beta 2^{\beta-1} \left(\frac{e_0}{g_n}\right)^{\beta-1} \sum_{k=2}^{n-1} \mathbf{1}_{\{H_{n,k}\}} \bar{T}_1^{(k)}\right] \\ &\leq \beta 2^{\beta-1} \mathbb{E}[e_0^{\beta-1}] (g_n)^{1-\beta} n^{(\alpha-1)(\beta-1)} \mathbb{E}[n^{\alpha-1} T_1^{(n)}] \\ &\leq \frac{K_3}{n^{\beta-1}} \leq \frac{K_3}{n}, \end{aligned}$$

where K_3 is a positive constant. We have used Lemma 2.1 to bound $\mathbb{E}[n^{\alpha-1}T_1^{(n)}]$.

Using (22),(23),(24),(27),(28), we have proved that for any $n, n \geq N$, if there exists $C > 0$ such that for all $2 \leq k \leq n-1$, $\mathbb{E}[(k^{\alpha-1}T_1^{(k)})^\beta] \leq C$, then

$$(29) \quad \mathbb{E}[(n^{\alpha-1}T_1^{(n)})^\beta] \leq \frac{C + \left(K_1 - C \frac{\alpha - (\alpha-1)\beta}{2(\alpha-1)} + C^{\frac{\beta-1}{\beta}} K_2 + K_3\right)}{n}.$$

Let C large enough such that

$$(30) \quad K_1 - C \frac{\alpha - (\alpha-1)\beta}{2(\alpha-1)} + C^{\frac{\beta-1}{\beta}} K_2 + K_3 < 0,$$

Then $\mathbb{E}[(n^{\alpha-1}T_1^{(n)})^\beta] \leq C$, which allows to conclude.

5. APPENDIX

A) The main recurrence tool

Lemma 5.1. *We consider the recurrence $a_n = b_n + \sum_{k=1}^{n-1} q_{n,k} a_k$. We assume that $b_n = o(n^{-1})$ and that there exist $\varepsilon > 0$ and $C > 0$ such that $1 - \sum_{k=1}^{n-1} q_{n,k} (\frac{n}{k})^\varepsilon \geq Cn^{-1}$ for n large enough. Then $\lim_{n \rightarrow +\infty} a_n = 0$.*

Proof. Let $(\bar{c}_n)_{n \geq 1}$ be an increasing sequence such that

$$\lim_{n \rightarrow +\infty} \bar{c}_n = +\infty; \quad \lim_{n \rightarrow +\infty} n b_n \bar{c}_n = 0.$$

Define another sequence $(c_n)_{n \geq 1}$ by: $c_1 = \bar{c}_1$. For $n \geq 1$,

$$c_{n+1} = \min\{c_n (\frac{n+1}{n})^\varepsilon, \bar{c}_{n+1}\},$$

Then we have $\lim_{n \rightarrow +\infty} c_n = +\infty, c_n b_n = o(n^{-1})$ and for any $1 \leq k \leq n-1$, $\frac{c_n}{c_k} \leq (\frac{n}{k})^\varepsilon$. In consequence, $1 - \sum_{k=1}^{n-1} q_{n,k} \frac{c_n}{c_k} \geq Cn^{-1}$ for n large enough. We suppose that there exist $n_1 > 0$ such that for $n > n_1$, we have $1 - \sum_{k=1}^{n-1} q_{n,k} \frac{c_n}{c_k} > \frac{C}{n}$ and $c_n b_n < \frac{C}{2n}$. We can find a number C' such that $C' > \max\{1, c_k a_k; 1 \leq k \leq n_1\}$. We transform the original recurrence to

$$c_n a_n = c_n b_n + \sum_{k=1}^{n-1} \left(q_{n,k} \frac{c_n}{c_k} \right) c_k a_k.$$

Then $c_{n_1+1} a_{n_1+1} \leq \frac{C}{2(n_1+1)} + (1 - \frac{C}{n_1+1}) C' \leq C'$. By induction, we prove that the sequence $(c_n a_n)_{n \geq 1}$ is bounded by C' . Since c_n tends to the infinity, we get $\lim_{n \rightarrow +\infty} a_n = 0$. \square

Remark 5.1. This kind of recurrence relationships is very frequent in probability. We refer to [34] for a rather detailed survey.

B) Asymptotic behaviours of $X_1^{(n)}$

Lemma 5.2. *We assume that $\rho(t)$ satisfies (4).*

$$(1) \text{ If } 0 < \zeta < \alpha - 1, \text{ then } \mathbb{E}[X_1^{(n)}] = \frac{1}{\alpha-1} \left(1 + \frac{C_1 \Gamma(2-\alpha+\zeta)\zeta}{C_0 \Gamma(2-\alpha)(\alpha-1-\zeta)} n^{-\zeta} \right) + o(n^{-\zeta}).$$

$$(2) \text{ If } \zeta = \alpha - 1, \text{ then } \mathbb{E}[X_1^{(n)}] = \frac{1}{\alpha-1} \left(1 + \frac{C_1(\alpha-1)}{C_0 \Gamma(2-\alpha)} n^{1-\alpha} \ln n \right) + o(n^{1-\alpha} \ln n).$$

$$(3) \text{ If } \zeta > \alpha - 1, \text{ then } \mathbb{E}[X_1^{(n)}] = \frac{1}{\alpha-1} \left(1 + \frac{C_2(\alpha-1)}{C_0 \Gamma(2-\alpha)} n^{1-\alpha} \right) + o(n^{1-\alpha}), \text{ where } C_2 = \lim_{t \rightarrow 0} \int_t^1 \rho(r) dr - \frac{C_0 t^{1-\alpha}}{\alpha-1}.$$

Proof. We have:

$$\mathbb{E}[X_1^{(n)}] = \frac{\int_0^1 (1-t)^{n-2} (\int_t^1 \rho(r) dr) dt}{\int_0^1 (1-t)^{n-2} t \rho(t) dt}$$

(see [13]). Hence we should give the values of $\int_0^1 (1-t)^{n-2} (\int_t^1 \rho(r) dr) dt$ and $\int_0^1 (1-t)^{n-2} t \rho(t) dt$. For that, we use the estimation (11).

(1) If $0 < \zeta < \alpha - 1$, $\int_t^1 \rho(r) dr = \int_t^1 C_0 r^{-\alpha} + C_1 r^{-\alpha+\zeta} + o(r^{-\alpha+\zeta}) dr = \frac{C_0}{\alpha-1} t^{1-\alpha} + \frac{C_1}{\alpha-1-\zeta} t^{-\alpha+\zeta+1} + o(t^{-\alpha+\zeta+1})$. Then using that $\alpha - \zeta - 1 > 0$, (11) and (13), we have

$$\begin{aligned} \int_0^1 (1-t)^{n-2} (\int_t^1 \rho(r) dr) dt &= \frac{C_0}{\alpha-1} \frac{\Gamma(n-1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} + (\frac{C_1}{\alpha-1-\zeta} + o(1)) \frac{\Gamma(n-1)\Gamma(2-\alpha+\zeta)}{\Gamma(n+1-\alpha+\zeta)} \\ &= \frac{C_0\Gamma(2-\alpha)}{\alpha-1} n^{\alpha-2} (1 + O(n^{-1})) + (\frac{C_1}{\alpha-1-\zeta} + o(1)) \Gamma(2-\alpha+\zeta) n^{\alpha-2-\zeta} (1 + O(n^{-1})) \\ &= \frac{C_0\Gamma(2-\alpha)}{\alpha-1} n^{\alpha-2} + \frac{C_1\Gamma(2-\alpha+\zeta)}{\alpha-1-\zeta} n^{\alpha-2-\zeta} + o(n^{\alpha-2-\zeta}), \end{aligned}$$

and

$$\begin{aligned} \int_0^1 (1-t)^{n-2} t \rho(t) dt &= C_0 \frac{\Gamma(n-1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} + (C_1 + o(1)) \frac{\Gamma(n-1)\Gamma(2-\alpha+\zeta)}{\Gamma(n+1-\alpha+\zeta)} \\ &= C_0 \Gamma(2-\alpha) n^{\alpha-2} (1 + O(n^{-1})) + (C_1 + o(1)) \Gamma(2-\alpha+\zeta) n^{\alpha-2-\zeta} (1 + O(n^{-1})) \\ &= C_0 \Gamma(2-\alpha) n^{\alpha-2} + C_1 \Gamma(2-\alpha+\zeta) n^{\alpha-2-\zeta} + o(n^{\alpha-2+\zeta}). \end{aligned}$$

Hence $\mathbb{E}[X_1^{(n)}] = \frac{1}{\alpha-1} (1 + \frac{C_1\Gamma(2-\alpha+\zeta)\zeta}{C_0\Gamma(2-\alpha)(\alpha-1-\zeta)} n^{-\zeta} + o(n^{-\zeta}))$.

(2) If $\zeta = \alpha - 1$, $\int_t^1 \rho(r) dr = \frac{C_0}{\alpha-1} t^{1-\alpha} - C_1 \ln t + o(\ln t)$. Then

$$\int_0^1 (1-t)^{n-2} (\int_t^1 \rho(r) dr) dt = \frac{C_0}{\alpha-1} \frac{\Gamma(n-1)\Gamma(2-\alpha)}{\Gamma(n+1-\alpha)} + (-C_1 + o(1)) \int_0^1 (1-t)^{n-2} \ln t dt.$$

Let $a_n = \int_0^1 (1-t)^n \ln t dt$. By integration by parts, $(n+1)a_n = na_{n-1} - \frac{1}{n+1}$. So $a_n = \frac{-\sum_{i=2}^{n+1} 1/i}{n+1} = \frac{-\ln n}{n} + O(n^{-1})$.

In consequence,

$$\begin{aligned} \int_0^1 (1-t)^{n-2} (\int_t^1 \rho(r) dr) dt &= \frac{C_0\Gamma(2-\alpha)}{\alpha-1} n^{\alpha-2} (1 + O(n^{-1})) + (-C_1 + o(1)) (\frac{-\ln(n-2)}{n-2} + O(n^{-1})) \\ &= \frac{C_0\Gamma(2-\alpha)}{\alpha-1} n^{\alpha-2} + C_1 \frac{\ln n}{n} + o(\frac{\ln n}{n}). \end{aligned}$$

Moreover,

$$\int_0^1 (1-t)^{n-2} t \rho(t) dt = C_0 \Gamma(2-\alpha) n^{\alpha-2} + C_1 n^{-1} + o(n^{-1}).$$

Hence, $\mathbb{E}[X_1^{(n)}] = \frac{1}{\alpha-1} (1 + \frac{C_1(\alpha-1)}{C_0\Gamma(2-\alpha)} n^{1-\alpha} \ln n) + o(n^{1-\alpha} \ln n)$.

(3) If $\zeta > \alpha - 1$, $\int_t^1 \rho(r)dr = \frac{C_0}{\alpha-1}t^{1-\alpha} + C_2 + o(1)$, where we recall that $C_2 = \lim_{t \rightarrow 0} \int_t^1 \rho(r)dr - \frac{C_0 t^{1-\alpha}}{\alpha-1}$. Notice that for the term $o(1)$ here, $\int_0^1 (1-t)^{n-2} o(1)dt = o(n^{-1})$. Hence,

$$\int_0^1 (1-t)^{n-2} \left(\int_t^1 \rho(r)dr \right) dt = \frac{C_0 \Gamma(2-\alpha)}{\alpha-1} n^{\alpha-2} + \frac{C_2}{n} + o(n^{-1}),$$

and,

$$\int_0^1 (1-t)^{n-2} t \rho(t) dt = C_0 \Gamma(2-\alpha) n^{\alpha-2} + O(n^{\alpha-2+\max\{-1, -\zeta\}}).$$

Hence we get $\mathbb{E}[X_1^{(n)}] = \frac{1}{\alpha-1} (1 + \frac{C_2(\alpha-1)}{C_0 \Gamma(2-\alpha)} n^{1-\alpha} + o(n^{1-\alpha}))$.

□

Lemma 5.3. *We assume that $\rho(t)$ satisfies (14). If $k \geq 2$,*

$$\mathbb{E}\left[\left(\frac{X_1^{(n)}}{n}\right)^k\right] = \frac{\int_0^1 k t^{k-1} \rho(t) dt}{C_0 \Gamma(2-\alpha)} n^{-\alpha} + O(n^{-\min\{1+\alpha, k\}}).$$

Proof. Let $B_{n,x}$ denote a binomial random variable with parameter (n, x) , $n \geq 2, 0 \leq x \leq 1$. Recall that for $2 \leq i \leq n$, $\mathbb{P}(X_1^{(n)} = i-1) = \int_0^1 \binom{n}{i} x^i (1-x)^{n-i} \nu(dx) / g_n = \int_0^1 \mathbb{P}(B_{n,x} = i) \nu(dx) / g_n$.

$$\begin{aligned} \mathbb{E}\left[\left(\frac{X_1^{(n)}}{n}\right)^k\right] &= \int_0^1 \mathbb{E}\left[\left(\frac{B_{n,x}-1}{n}\right)^k \mathbf{1}_{B_{n,x} \geq 1}\right] \nu(dx) / g_n \\ &= \int_0^1 n^{-k} \mathbb{E}[(B_{n,x}^k - B_{n,x}) \\ &\quad + \sum_{i=1}^{k-1} \binom{k}{i} (-1)^i (B_{n,x}^{k-i} - B_{n,x}) + (-1)^k (1 - B_{n,x}) \mathbf{1}_{B_{n,x} \geq 1}] \nu(dx) / g_n. \end{aligned}$$

Using Lemma 5.5 in Appendix C, we get $\mathbb{E}[B_{n,x}^k - B_{n,x}] = (nx)^k + O(n^{k-1})x^2$. Then

$$\begin{aligned} \mathbb{E}\left[\left(\frac{X_1^{(n)}}{n}\right)^k\right] &= \int_0^1 n^{-k} ((nx)^k + O(n^{k-1})x^2) \nu(dx) / g_n + n^{-k} \int_0^1 (-1)^k (1 - nx - (1-x)^n) \nu(dx) / g_n \\ &= \frac{\int_0^1 x^k \nu(dx) n^{-\alpha}}{C_0 \Gamma(2-\alpha)} + O(n^{-\min\{1+\alpha, k\}}) = \frac{\int_0^1 k t^{k-1} \rho(t) dt n^{-\alpha}}{C_0 \Gamma(2-\alpha)} + O(n^{-\min\{1+\alpha, k\}}). \end{aligned}$$

In the second equality, we have used $g_n \sim C_0 \Gamma(2-\alpha) n^\alpha$ and also the fact that $\int_0^1 (1 - nx - (1-x)^n) \nu(dx) \leq g_n = \int_0^1 (1 - nx(1-x)^{n-1} - (1-x)^n) \nu(dx)$. This achieves the proof.

□

Proposition 5.4. *We assume that $\rho(t)$ satisfies (4) and $r \in [0, \omega)$.*

- (1) *If $0 < \zeta < \alpha - 1$, then $\mathbb{E}\left[\left(\frac{n}{n-X_1^{(n)}}\right)^r\right] = 1 + \frac{r}{n(\alpha-1)} + \frac{r C_1 \Gamma(2-\alpha+\zeta) \zeta}{C_0 (\alpha-1) \Gamma(2-\alpha) (\alpha-1-\zeta)} n^{-\zeta-1} + o(n^{-\zeta-1})$.*
- (2) *If $\zeta = \alpha - 1$, then $\mathbb{E}\left[\left(\frac{n}{n-X_1^{(n)}}\right)^r\right] = 1 + \frac{r}{n(\alpha-1)} + \frac{r C_1}{C_0 \Gamma(2-\alpha)} n^{-\alpha} \ln n + o(n^{-\alpha} \ln n)$.*
- (3) *If $\zeta > \alpha - 1$, then $\mathbb{E}\left[\left(\frac{n}{n-X_1^{(n)}}\right)^r\right] = 1 + \frac{r}{n(\alpha-1)} + \left(\frac{\int_0^1 ((1-x)^{-r} - 1 - rx) \nu(dx)}{C_0 \Gamma(2-\alpha)} + \frac{r C_2}{C_0 \Gamma(2-\alpha)}\right) n^{-\alpha} + o(n^{-\alpha})$, where C_2 is the same as in Theorem 2.1.*

Proof. By Taylor expansion formula, for $m \geq 2$, we have,

$$\begin{aligned} \left(\frac{n}{n - X_1^{(n)}} \right)^r &= \left(\frac{1}{1 - \frac{X_1^{(n)}}{n}} \right)^r \\ &= 1 + r \frac{X_1^{(n)}}{n} + \sum_{k=2}^m \frac{\prod_{i=0}^{k-1} (r+i)}{k!} \left(\frac{X_1^{(n)}}{n} \right)^k + \frac{\prod_{i=0}^m (r+i)}{m!} \int_0^{\frac{X_1^{(n)}}{n}} (1-t)^{-r-m-1} \left(\frac{X_1^{(n)}}{n} - t \right)^m dt. \end{aligned}$$

Then we get:

(31)

$$\mathbb{E} \left[\left(\frac{n}{n - X_1^{(n)}} \right)^r \right] = 1 + r \mathbb{E} \left[\frac{X_1^{(n)}}{n} \right] + \sum_{k=2}^m \frac{\prod_{i=0}^{k-1} (r+i)}{k!} \mathbb{E} \left[\left(\frac{X_1^{(n)}}{n} \right)^k \right] + \frac{\prod_{i=0}^m (r+i)}{m!} \mathbb{E} \left[\int_0^{\frac{X_1^{(n)}}{n}} (1-t)^{-r-m-1} \left(\frac{X_1^{(n)}}{n} - t \right)^m dt \right].$$

Thanks to Lemma 5.2 and 5.3, we have the asymptotic behaviours of $\mathbb{E} \left[\frac{X_1^{(n)}}{n} \right]$ and $\mathbb{E} \left[\sum_{k=2}^m \frac{\prod_{i=0}^{k-1} (r+i)}{k!} \left(\frac{X_1^{(n)}}{n} \right)^k \right]$. In particular, for $m \geq 2$,

$$\lim_{n \rightarrow +\infty} n^\alpha \mathbb{E} \left[\sum_{k=2}^m \frac{\prod_{i=0}^{k-1} (r+i)}{k!} \left(\frac{X_1^{(n)}}{n} \right)^k \right] = \sum_{k=2}^m \frac{\prod_{i=0}^{k-1} (r+i)}{k!} \frac{\int_0^1 kx^{k-1} \rho(t) dt}{C_0 \Gamma(2-\alpha)} = \sum_{k=2}^m \frac{\prod_{i=0}^{k-1} (r+i)}{k!} \frac{\int_0^1 x^k \nu(x)}{C_0 \Gamma(2-\alpha)}.$$

In consequence,

$$\lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} n^\alpha \mathbb{E} \left[\sum_{k=2}^m \frac{\prod_{i=0}^{k-1} (r+i)}{k!} \left(\frac{X_1^{(n)}}{n} \right)^k \right] = \lim_{m \rightarrow +\infty} \sum_{k=2}^m \frac{\prod_{i=0}^{k-1} (r+i)}{k!} \frac{\int_0^1 x^k \nu(x)}{C_0 \Gamma(2-\alpha)} = \frac{\int_0^1 ((1-x)^{-r} - 1 - rx) \nu(dx)}{C_0 \Gamma(2-\alpha)}.$$

we need only to estimate $\frac{\prod_{i=0}^m (r+i)}{m!} \mathbb{E} \left[\int_0^{\frac{X_1^{(n)}}{n}} (1-t)^{-r-m-1} \left(\frac{X_1^{(n)}}{n} - t \right)^m dt \right]$, which is the sum of two terms $P_1(m, n, s)$ and $P_2(m, n, s)$ with $0 < s < 1$ and

$$P_1(m, n, s) = \frac{\prod_{i=0}^m (r+i)}{m!} \mathbb{E} \left[\int_0^{\frac{X_1^{(n)}}{n}} (1-t)^{-r-m-1} \left(\frac{X_1^{(n)}}{n} - t \right)^m dt \mathbf{1}_{X_1^{(n)} \geq ns} \right],$$

$$P_2(m, n, s) = \frac{\prod_{i=0}^m (r+i)}{m!} \mathbb{E} \left[\int_0^{\frac{X_1^{(n)}}{n}} (1-t)^{-r-m-1} \left(\frac{X_1^{(n)}}{n} - t \right)^m dt \mathbf{1}_{X_1^{(n)} < ns} \right].$$

We first focus on $P_1(m, n, s)$. By Lemma 5.7 in Appendix C, we have

$$(32) \quad P_1(m, n, s) \leq \mathbb{E} \left[\left(\frac{n}{n - X_1^{(n)}} \right)^r \mathbf{1}_{X_1^{(n)} \geq ns} \right] \leq n^{-\alpha} K_4 s^{-\alpha} (1-s)^{\bar{r}-r},$$

where $\bar{r} \in (r, \omega)$ and K_4 is a number depending only on \bar{r} and ν (it is important to notice that it does not depend on s).

We now give an upper bound for $P_2(m, n, s)$. We have

$$n^\alpha P_2(m, n, s) = n^\alpha \frac{\prod_{i=0}^m (r+i)}{m!} \mathbb{E} \left[\int_0^{\frac{X_1^{(n)}}{n}} (1-t)^{-r-1} \left(\frac{X_1^{(n)}/n - t}{1-t} \right)^m dt \mathbf{1}_{X_1^{(n)} < ns} \right].$$

For $t \in [0, x)$ with $0 < x \leq 1$, we have:

$$\frac{x-t}{1-t} \leq x.$$

So we deduce that

$$\int_0^{\frac{X_1^{(n)}}{n}} \left(\frac{X_1^{(n)}/n - t}{1-t} \right)^m dt \leq \left(\frac{X_1^{(n)}}{n} \right)^{m+1}.$$

Hence, using Lemma 5.3, for $m > 2$,

$$\begin{aligned} n^\alpha P_2(m, n, s) &\leq n^\alpha \frac{\prod_{i=0}^m (r+i)}{m!} (1-s)^{-r-1} \mathbb{E}[(X_1^{(n)}/n)^{m+1}] \\ &= (1-s)^{-r-1} \frac{\prod_{i=0}^m (r+i)}{m!} \left(\frac{\int_0^1 (m+1)t^m \rho(t) dt}{C_0 \Gamma(2-\alpha)} + O(n^{-1}) \right). \end{aligned}$$

Using Lemme 5.6 in Appendix C, we have

$$\begin{aligned} \int_0^1 (m+1)t^m \rho(t) dt &= \int_0^1 x^{m+1} \nu(dx) \\ &= \int_0^1 x^{m+1} (1-x)^{\bar{r}} \frac{\nu(dx)}{(1-x)^{\bar{r}}} \\ &= \int_0^1 x^{m+1} (1-x)^{\bar{r}} \nu^{(-\bar{r})}(dx) \leq K_5 m^{-\bar{r}}, \end{aligned}$$

where K_5 is a positive real number depending only on \bar{r} and ν .

Notice that $\frac{\prod_{i=0}^m (r+i)}{m!} = \frac{\Gamma(m+r+1)}{\Gamma(r)\Gamma(m+1)} \sim \frac{m^r}{\Gamma(r)}$. Hence

$$(33) \quad P_2(m, n, s) \leq n^{-\alpha} (1-s)^{-r-1} m^r (O(m^{-\bar{r}}) + o(n^{-1})).$$

Combining (32) and (33), we deduce that

$$\lim_{m \rightarrow +\infty} \limsup_{n \rightarrow +\infty} n^\alpha (P_1(m, n, s) + P_2(m, n, s)) = \lim_{m \rightarrow +\infty} \limsup_{n \rightarrow +\infty} n^\alpha \frac{\prod_{i=0}^m (r+i)}{m!} \mathbb{E} \left[\int_0^{\frac{X_1^{(n)}}{n}} (1-t)^{-r-m-1} \left(\frac{X_1^{(n)}}{n} - t \right)^m dt \right] = 0.$$

This convergence together with Lemma 5.2 and 5.3 yield this proposition. \square

Remark 5.2. Using the same arguments, it is easy to prove that if $l \in \mathbb{N}$, $\mathbb{E}[(\frac{n}{n-X_1^{(n-l)}})^r]$ has also the decomposition given by Proposition 5.4. More precisely, write

$$\left(\frac{n}{n-X_1^{(n-l)}} \right)^r = 1 + r \frac{X_1^{(n-l)}}{n} + \sum_{k=2}^{+\infty} \frac{\prod_{i=0}^{k-1} (r+i)}{k!} \left(\frac{X_1^{(n-l)}}{n} \right)^k.$$

For any $k \geq 1$, Lemma 5.2 and 5.3 give $\mathbb{E}[(\frac{X_1^{(n-l)}}{n})^k] = \mathbb{E}[(\frac{X_1^{(n-l)}}{n-l})^k] + O(n^{-2})$. This proposition also shows that

$$n^\alpha \mathbb{E} \left[\sum_{k=2}^{+\infty} \frac{\prod_{i=0}^{k-1} (r+i)}{k!} \left(\frac{X_1^{(n-l)}}{n} \right)^k \right] \leq n^\alpha \mathbb{E} \left[\sum_{k=2}^{+\infty} \frac{\prod_{i=0}^{k-1} (r+i)}{k!} \left(\frac{X_1^{(n-l)}}{n-l} \right)^k \right] \xrightarrow{m \rightarrow +\infty} 0.$$

Then we can conclude.

Remark 5.3. If we move farther to Remark 5.4, we will see that in the case of Beta($2-\alpha, \alpha$), if $r \geq \omega$, we have

$$\lim_{m \rightarrow +\infty} \lim_{n \rightarrow +\infty} n^\alpha \frac{\prod_{i=0}^m (r+i)}{m!} \mathbb{E} \left[\int_0^{\frac{X_1^{(n)}}{n}} (1-t)^{-r-m-1} \left(\frac{X_1^{(n)}}{n} - t \right)^m dt \right] \geq C,$$

where $C > 0$ is defined in Remark 5.4. Then in the third case of this Proposition 5.4, the expression of $\mathbb{E}[(\frac{n}{n-X_1^{(n)}})^r]$ will be changed. So the constraint $r \in [0, \omega)$ is necessary.

C) Some results necessary to prove those in Appendix B

Lemma 5.5. *Let $B_{n,x}$ be a binomial random variable with parameter (n, x) , $n \geq 2, 0 \leq x \leq 1$. Let k be an integer such that $2 \leq k \leq n$. Then*

$$nx + n(n-1) \cdots (n-k+1)x^k \leq \mathbb{E}[B_{n,x}^k] \leq (nx)^k + \binom{k}{2} n^{k-1} x^2,$$

Proof. Write $B_{n,x} = Y_1 + \dots + Y_n$, where Y_1, \dots, Y_n are independent Bernoulli random variables. Let $S := \{\{i_1, \dots, i_k\}; 1 \leq i_1, \dots, i_k \leq n\}$. Then

$$\begin{aligned} \mathbb{E}\left[\sum_{\{i_1, \dots, i_k\} \in S_1} Y_{i_1} \dots Y_{i_k}\right] + \mathbb{E}\left[\sum_{\{i_1, \dots, i_k\} \in S_3} Y_{i_1} \dots Y_{i_k}\right] &\leq \mathbb{E}[(B_{n,x})^k] \\ &\leq \mathbb{E}\left[\sum_{\{i_1, \dots, i_k\} \in S_2} Y_{i_1} \dots Y_{i_k}\right] + \mathbb{E}\left[\sum_{\{i_1, \dots, i_k\} \in S_3} Y_{i_1} \dots Y_{i_k}\right], \end{aligned}$$

where

- (1) $S_1 := \{\{i_1, \dots, i_n\} \in A; i_1 = \dots = i_k\}$. Then $\mathbb{E}[\sum_{\{i_1, \dots, i_k\} \in S_1} Y_{i_1} \dots Y_{i_k}] = nx$.
- (2) $S_2 := \{\{i_1, \dots, i_n\} \in A; \exists 1 \leq p < q \leq k, i_p = i_q\}$. Then $\mathbb{E}[\sum_{\{i_1, \dots, i_k\} \in S_2} Y_{i_1} \dots Y_{i_k}] \leq \binom{k}{2} n^{k-1} x^2$.
- (3) $S_3 := \{\{i_1, \dots, i_n\} \in A; \forall 1 \leq p < q \leq k, i_p \neq i_q\}$. Then $\mathbb{E}[\sum_{\{i_1, \dots, i_k\} \in S_3} Y_{i_1} \dots Y_{i_k}] = n(n-1) \dots (n-k+1) x^k$.

Then we can conclude. \square

Lemma 5.6. *We assume that $\rho(t)$ satisfies condition (14). Then for every $s \geq 0$, $\int_0^1 x^n (1-x)^s \nu(dx) \leq K_6 n^{-s}$, where K_6 is a positive constant which depends only on s and ν .*

Proof. It is clear that there exists $K_7 > 0$ such that $\rho(t) \leq K_7 t^{-\alpha}$, for all $0 < t \leq 1$. Then

$$\begin{aligned} \int_0^1 x^n (1-x)^s \nu(dx) &= \int_0^1 \rho(t) (n - (n+s)t) t^{n-1} (1-t)^{s-1} dt \\ &\leq \int_0^1 \rho(t) (n - nt) t^{n-1} (1-t)^{s-1} dt \\ &\leq n K_7 \int_0^1 t^{n-1-\alpha} (1-t)^s dt = n K_7 \frac{\Gamma(n-\alpha) \Gamma(s+1)}{\Gamma(n-\alpha+s+1)} \leq K_6 n^{-s}, \end{aligned}$$

for some K_6 which only depends on K_7 and s . This achieves the proof of the lemma. \square

The upper bound of $P_1(m, n, s)$ is given by the next lemma.

Lemma 5.7. *We assume that $\rho(t)$ satisfies condition (14). Let $r \in [0, \omega)$ and $\bar{r} \in (r, \omega)$. Then there exists a constant K_{11} depending only on \bar{r} and ν such that for all $s \in (0, 1)$,*

$$P_1(m, n, s) \leq \mathbb{E}\left[\left(\frac{n}{n - X_1^{(n)}}\right)^r \mathbf{1}_{X_1^{(n)} \geq ns}\right] \leq n^{-\alpha} K_{11} s^{-\alpha} (1-s)^{\bar{r}-r}.$$

Proof. It is easy to observe that

$$P_1(m, n, s) \leq \mathbb{E}\left[\left(\frac{n}{n - X_1^{(n)}}\right)^r \mathbf{1}_{X_1^{(n)} \geq ns}\right].$$

For $x \in \mathbb{R}$, we define $\lceil x \rceil = \min\{m \in \mathbb{Z}; m \geq x\}$. We have

$$\mathbb{E}\left[\left(\frac{n}{n - X_1^{(n)}}\right)^r \mathbf{1}_{X_1^{(n)} \geq ns}\right] = \sum_{k=\lceil ns \rceil}^{n-1} \frac{\int_0^1 \binom{n}{k+1} x^{k+1} (1-x)^{n-k-1} \left(\frac{n}{n-k}\right)^r \nu(dx)}{g_n}.$$

Notice that $\binom{n}{k+1} \left(\frac{n}{n-k}\right)^r = \frac{\Gamma(n+1)}{\Gamma(k+2)\Gamma(n-k)} \left(\frac{n}{n-k}\right)^r$. So by (11), there exist two positive constants K_8, K_9 such that for all $k \in \{1, 2, \dots, n-1\}$,

$$K_8 \frac{\Gamma(n+1+r)}{\Gamma(k+2)\Gamma(n-k+r)} \leq \left(\frac{n}{k+1}\right) \left(\frac{n}{n-k}\right)^r \leq K_9 \frac{\Gamma(n+1+r)}{\Gamma(k+2)\Gamma(n-k+r)}.$$

Moreover using integration by parts, for $1 \leq l \leq n-1$ and $0 \leq x \leq 1$, we have:

$$\begin{aligned}
 (34) \quad & \sum_{k=l}^{n-1} \frac{\Gamma(n+1+r)}{\Gamma(k+2)\Gamma(n-k+r)} x^{k+1}(1-x)^{n-k-1+r} \\
 &= \frac{\Gamma(n+1+r)}{\Gamma(l+1)\Gamma(n-l+r)} \int_0^x t^l(1-t)^{n-l+r-1} dt + \frac{\Gamma(n+1+r)}{\Gamma(n+1)\Gamma(1+r)} x^n(1-x)^r \\
 &\quad - \frac{\Gamma(n+1+r)}{\Gamma(n)\Gamma(1+r)} \int_0^x t^{n-1}(1-t)^r dt.
 \end{aligned}$$

Let $\bar{r} \in (r, \omega)$, $\nu^{(-\bar{r})}(dx) = \frac{\nu(dx)}{(1-x)^{\bar{r}}}$, and $\rho^{(-\bar{r})}(t) = \int_t^1 \nu^{(-\bar{r})}(r) dr$. It is easy to see that $\rho^{(-\bar{r})}(t) = C_0 t^{-\alpha} + o(t^{-\alpha})$. Then there exists $K_{10} > 0$, such that $\rho^{(-\bar{r})}(t) \leq K_{10} t^{-\alpha}$ for all $t \in (0, 1]$.

$$\begin{aligned}
 & \mathbb{E} \left[\left(\frac{n}{n - X_1^{(n)}} \right)^{\bar{r}} \mathbf{1}_{X_1^{(n)} \geq ns} \right] \\
 &= \sum_{k=\lceil ns \rceil}^{n-1} \frac{\int_0^1 \binom{n}{k+1} x^{k+1} (1-x)^{n-k-1} \left(\frac{n}{n-k} \right)^{\bar{r}} \nu(dx)}{g_n} = \sum_{k=\lceil ns \rceil}^{n-1} \frac{\int_0^1 \binom{n}{k+1} x^{k+1} (1-x)^{n-k-1+\bar{r}} \left(\frac{n}{n-k} \right)^{\bar{r}} \nu^{(-\bar{r})}(dx)}{g_n} \\
 &\leq K_9 \frac{\int_0^1 \frac{\Gamma(n+1+\bar{r})}{\Gamma(\lceil ns \rceil+1)\Gamma(n-\lceil ns \rceil+\bar{r})} \int_0^x t^{\lceil ns \rceil} (1-t)^{n-\lceil ns \rceil+\bar{r}-1} dt \nu^{(-\bar{r})}(dx)}{g_n} + K_9 \frac{\int_0^1 \frac{\Gamma(n+1+\bar{r})}{\Gamma(n+1)\Gamma(1+\bar{r})} x^n (1-x)^{\bar{r}} \nu^{(-\bar{r})}(dx)}{g_n} \\
 &\leq K_9 \frac{\int_0^1 \frac{\Gamma(n+1+\bar{r})}{\Gamma(\lceil ns \rceil+1)\Gamma(n-\lceil ns \rceil+\bar{r})} \rho^{(-\bar{r})}(x) x^{\lceil ns \rceil} (1-x)^{n-\lceil ns \rceil+\bar{r}-1} dx}{g_n} + K_9 \frac{\int_0^1 \frac{\Gamma(n+1+\bar{r})}{\Gamma(n+1)\Gamma(1+\bar{r})} x^n (1-x)^{\bar{r}} \nu^{(-\bar{r})}(dx)}{g_n} \\
 &\leq K_9 K_{10} \frac{\frac{\Gamma(n+1+\bar{r})\Gamma(\lceil ns \rceil+1-\alpha)}{\Gamma(\lceil ns \rceil+1)\Gamma(n+1+\bar{r}-\alpha)}}{g_n} + K_6 K_9 \frac{\frac{\Gamma(n+1+\bar{r})}{\Gamma(n+1)\Gamma(1+\bar{r})} n^{-\bar{r}}}{g_n} \\
 &\leq K_{11} s^{-\alpha} n^{-\alpha},
 \end{aligned}$$

where for the first inequality, we use (34) with $l = \lceil ns \rceil$, in the second inequality, we have used an argument of integration by parts and for the third inequality, we bound $\rho^{(-\bar{r})}(x)$ by $K_{10} x^{-\alpha}$ and we also use Lemma 5.6. For the last inequality, we use (11). Here K_{11} is a constant which depends only on \bar{r} and ν . Hence for all $n \geq 2$,

$$n^\alpha \mathbb{E} \left[\left(\frac{n}{n - X_1^{(n)}} \right)^{\bar{r}} \mathbf{1}_{X_1^{(n)} \geq ns} \right] \leq n^\alpha (1-s)^{\bar{r}-r} \mathbb{E} \left[\left(\frac{n}{n - X_1^{(n)}} \right)^{\bar{r}} \mathbf{1}_{X_1^{(n)} \geq ns} \right] \leq K_{11} s^{-\alpha} (1-s)^{\bar{r}-r}.$$

This achieves the proof of the lemma. \square

Remark 5.4. If $r \geq \omega$, this lemma is false. Take $\text{Beta}(2-\alpha, \alpha)$ as example. We have $\nu(dx) = \frac{1}{\Gamma(\alpha)\Gamma(2-\alpha)} x^{-1-\alpha} (1-x)^{\alpha-1} dx$ and $\omega = \alpha$. Then for any fixed $0 < s < 1$ and $n \geq \frac{1}{1-s}$, we have

$ns \geq n - 1$ and it follows that

$$\begin{aligned}
P_1(m, n, s) &\geq \mathbb{E}\left[\left(\left(\frac{n}{n - X_1^{(n)}}\right)^r - 1 - r\frac{X_1^{(n)}}{n} - \sum_{k=2}^m \frac{\prod_{i=0}^{k-1}(r+i)}{k!} \left(\frac{X_1^{(n)}}{n}\right)^k\right) \mathbf{1}_{X_1^{(n)}=n-1}\right] \\
&= \mathbb{P}(X_1^{(n)} = n - 1) \left(n^r - 1 - r\frac{n-1}{n} - \sum_{k=2}^m \frac{\prod_{i=0}^{k-1}(r+i)}{k!} \left(\frac{n-1}{n}\right)^k\right) \\
&= \frac{\int_0^1 x^n \nu(dx)}{g_n} \left(n^r - 1 - r\frac{n-1}{n} - \sum_{k=2}^m \frac{\prod_{i=0}^{k-1}(r+i)}{k!} \left(\frac{n-1}{n}\right)^k\right) \\
&\sim Cn^{-2\alpha} \left(n^r - 1 - r\frac{n-1}{n} - \sum_{k=2}^m \frac{\prod_{i=0}^{k-1}(r+i)}{k!} \left(\frac{n-1}{n}\right)^k\right)
\end{aligned}$$

where C is a strictly positive number. Hence, if $r \geq \omega = \alpha$, then

$$\liminf_{n \rightarrow +\infty} n^\alpha P_1(m, n, s) \geq C, \forall 0 < s < 1.$$

This result is not compatible with the lemma with $0 \leq r < \omega$. This remark justifies the constraint $0 \leq r < \omega$.

D) Results that are used to prove Theorem 1.3.

Lemma 5.8. *Let $a > 0, b > 0, \beta > 2$. Then*

$$0 < (a + b)^\beta \leq a^\beta + b^\beta + \beta 2^{\beta-1} ab^{\beta-1} + \beta 2^{\beta-1} ba^{\beta-1}.$$

Proof. If $0 \leq m \leq 1$, then

$$(1 + m)^\beta \leq 1 + \beta 2^{\beta-1} m \leq 1 + m^\beta + \beta 2^{\beta-1} m + \beta 2^{\beta-1} m^{\beta-1}.$$

We use that the function $m \mapsto (1 + m)^\beta$ is convex and that $\beta 2^{\beta-1}$ is the derivative of $(1 + m)^\beta$ at $m = 1$.

If $1 < m$, then

$$(1 + m)^\beta = m^\beta \left(1 + \frac{1}{m}\right)^\beta \leq (m)^\beta \left(1 + \beta 2^{\beta-1} \frac{1}{m}\right) \leq 1 + m^\beta + \beta 2^{\beta-1} m + \beta 2^{\beta-1} m^{\beta-1}.$$

Hence for all $m > 0$,

$$(1 + m)^\beta \leq 1 + m^\beta + \beta 2^{\beta-1} m + \beta 2^{\beta-1} m^{\beta-1}.$$

Then for all $a > 0, b > 0$,

$$\begin{aligned}
(a + b)^\beta &= a^\beta \left(1 + \frac{b}{a}\right)^\beta \leq a^\beta \left(1 + \left(\frac{b}{a}\right)^\beta + \beta 2^{\beta-1} \frac{b}{a} + \beta 2^{\beta-1} \left(\frac{b}{a}\right)^{\beta-1}\right) \\
&= a^\beta + b^\beta + \beta 2^{\beta-1} ab^{\beta-1} + \beta 2^{\beta-1} ba^{\beta-1}.
\end{aligned}$$

This achieves the proof. □

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